Journal of Statistical Physics, Vol. 76, Nos. 3/4, 1994

A Model of Modulated Diffusion. I. Analytical Results

A. Bazzani,[1] S. Siboni,[2,3] G. Turchetti,[2] and S. Vaienti[3]

Received August 27, 1993; final March 14, 1994

We introduce an integrable isochronous system and perturb its fréquency by an external—deterministic or purely random—noise. Under the perturbation the action variable evolves in time: the corresponding diffusion coefficient is exactly computed and its dependence on the magnitude of the perturbation is carefully investigated. Different behaviors are found and justified: the quasilinear approximation, the superlinear regime, and the ballistic motion.

KEY WORDS: Modulated diffusion; standard mappings; random phase approximation.

1. INTRODUCTION

Transport in chaotic regions of phase space is an open problem for Hamiltonian systems. For a confined hot plasma, a beam in a magnetic lattice, or a spinning planet it is very likely the most important phenomenon which determines the lifetime of the system.(1-8) The statistical properties of transport are far from being understood, owing to the complexity and variety of topological structures in phase space: in the simplest case of two-dimensional symplectic mappings, invariant curves, cantori, chains of islands, and hyperbolic manifolds and their replicas under scale changes coexist and make the patterns of transport highly complex. Intensive investigations have been carried out for two-dimensional systems such as smooth symplectic mappings with the twist property (Chirikov–Greene standard map), but a complete description is available only for a class of

[1] Istituto di Matematica dell'Università Bologna and I.N.F.N. Sezione Bologna, Bologna, Italy.

[2] Istituto di Fisica dell'Università Bologna and I.N.F.N. Sezione Bologna, Bologna, Italy.

[3] Centre de Physique Théorique, C.N.R.S., Luminy, Marseille, France.

almost hyperbolic piecewise linear mappings of the torus lifted to the cylinder,[7-12] whose behavior is very close to billiards.[13-16] In fact in these cases the statistical properties (decay of correlations, central limit theorem, and invariance principle) are provided by the construction of a (possibly infinite) Markov partition which allows one to use the powerful tool of symbolic dynamics or, more generally, of the theory of the denumerable Markov chains and the Perron–Frobenius theory.[17,18] Here we propose a model which gives a description of the local diffusion properties of a symplectic map when an external modulation is introduced. Such a model is physically motivated by the dynamics of a beam whose linear frequency is modulated either periodically or stochastically owing to low Fourier components or noisy fluctuations of the current or by the coupling with the hyperbolic component of another degree of freedom. The heating of a plasma by an electromagnetic wave seems also to give rise to a similar description locally in phase space. Finally the Arnol'd diffusion of the spin–orbit coupling in the motion of a planet has its root in the coupling of a hyperbolic degree of freedom with a regular integrable motion.[19-21] To this end we have investigated four types of modulations[22,23],[4]:

(i) The stochastic modulation given by an i.i.d. random process whose distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

(ii) The Bernoulli shift on a space of symbols of finite cardinality.

(iii) The Markov map $T(\alpha)=2\alpha \bmod [0,1[-\frac{1}{2}$, $\alpha\in[-\frac{1}{2},\frac{1}{2}[$.

(iv) A periodic or quasiperiodic modulation.

In all these models analytical solutions are found for the diffusion coefficient defined as the limit of the mean square deviation of the action $(j_n-j_0)^2$ averaged on the initial angle and the measure space for the process divided by $2n$, namely

$$D=\lim_{n\to+\infty}\frac{1}{2n}\cdot\mathbf{E}((j_n-j_0)^2)$$

We will actually use a slightly different definition, which is nevertheless equivalent to that given before; we will also show that the diffusion coefficient just defined is independent of j_0 and this explains why we dropped the dependence on the initial action. The behavior of the process (iv) is distinct from the others: indeed, if a nonresonance Diophantine condition is satisfied by the frequencies, the existence of two-dimensional tori $\mathbb{T}^2$ for the

[4] The same perturbations but stochastic ones are also considered in the recent investigation of a quantum-like model.[24]

associated autonomous volume-preserving map is a topological barrier to diffusion, and $D=0$. In all the other cases, depending upon some technical conditions which will be discussed later, the diffusion coefficient exists, can be expressed in a finite form, and has the following asymptotic behavior. We also produce an example where for large values of the perturbation parameter ε, the diffusion coefficient does not converge to its quasilinear estimate. Instead, for all the other models, when $\varepsilon \to \infty$ the quasilinear value is obtained, which corresponds to the random phase approximation of the angles. For intermediate values of ε the behavior of D sharply depends on the process. In particular D can be larger than the quasilinear value and exhibit monotonic or oscillatory approach to the asymptotic limits. Such behaviors are indeed observed in numerical simulations of Lagrangian diffusion of test particles in a plasma in the so-called electrostatic model.[25] For small modulation amplitudes the diffusion coefficient turns out to behave as ε^2 as $\varepsilon \to 0$. We also produce an example where the diffusion coefficient is infinite, due to the ballistic motion of the action variable.

In the second part of this work[26] the complete statistical properties are explored in order to prove the possible existence of a central limit theorem and invariance principle. The plan of the paper is the following: in Section 2 we present the class of models we deal with and state the principal results. The proof for the main result is reported in Section 3, whereas Section 4 is devoted to comments and conclusions.

2. THE MODEL AND STATEMENT OF THE RESULTS

2.1. The Model

We first introduce an integrable area-preserving map on the cylinder $\{(\theta, j) \in \mathbb{T}^1 \times \mathbb{R}\}$, where, with abuse of language, we put $\mathbb{T}^1 = [0, 2\pi[$ the parametrization of the torus:

$$M\colon\quad \begin{cases} \theta' = \theta + \omega \quad \mathrm{mod}[0, 2\pi[\\ j' = j + V(\theta) \end{cases} \tag{2.1.1}$$

where $V(\theta)$ is an analytic periodic function of period 2π and zero mean, and $\omega/2\pi$ *is a Diophantine number*. The motion takes place on invariant curves characterized by the same rotation number ω; these invariant curves will be parametrized by the equation $J = G(\theta, j)$, $J \in \mathbb{R}$. Starting in Section 2.2, it will be convenient to work with the coordinates (θ, J): the change of variable is obtained by a straightforward calculation, which uses the exponential decay of the Fourier coefficients V_k and the Diophantine hypothesis on the rotation number $\omega/2\pi$:

$$J = G(\theta, j) = j - F(\theta) = j - \sum_{k=-\infty}^{k=+\infty}{}' \frac{V_k}{e^{ik\omega} - 1} e^{ik\theta} \tag{2.1.2}$$

The system (2.1.1) is clearly an approximation of those anisochronous Hamiltonian systems where the frequency varies adiabatically with the action in some region S of the phase space.

We now introduce a perturbation in the evolution of the θ variable that will be of two types: deterministic or purely stochastic. Numerical simulations suggest that the behavior of these perturbed models is quite similar to that of more complex weakly anisochronous systems.

2.1.1. Deterministic Perturbation. In this case we couple the map M with another transformation $\alpha' = T(\alpha)$ which will be defined later in such a way that our original system is replaced by

$$M_p: \quad \begin{cases} \alpha' = T(\alpha) \\ \theta' = \theta + \omega + f(\alpha)\varepsilon \quad \operatorname{mod}[0, 2\pi[\\ j' = j + V(\theta) \end{cases} \tag{2.1.3}$$

where from now on ε is a real number different from zero.

The mapping T and the function f will be chosen in one of the following forms:

1. $T(\alpha) = 2\alpha \operatorname{mod}[0, 1[- \frac{1}{2}$, $\alpha \in [-\frac{1}{2}, \frac{1}{2}[$, which is a Markovian map of the unit circle with respect to the Lebesgue measure μ_L; and $f(\alpha) = \alpha$, $\forall \alpha \in [-\frac{1}{2}, \frac{1}{2}[$, periodically continued with period 1.

2. $T(\alpha) = \sigma(\alpha)$, where $\alpha = \{\alpha_i\}_{i=-\infty}^{\infty}$ belongs to the space A of bi-infinite sequences at values in the set $X = \{s_1, ..., s_N\}$, $s_i \in \mathbb{R}$, $1 \leqslant i \leqslant N$, endowed with the discrete topology. The set of probabilities $\mu\{s_i\} = p_i$, $0 < p_i < 1$, $\sum_{i=1}^{N} p_i = 1$, defines a measure $\bar{\mu}$ on A through

$$\bar{\mu} = \bigotimes_{i=-\infty}^{\infty} \mu \tag{2.1.4}$$

This measure is invariant with respect to the transformation σ acting on A as the shift $(\sigma\alpha)_i = \alpha_{i+1}$. The shift automorphism σ with the invariant measure $\bar{\mu}$ is called a Bernoulli automorphism on the probability space $(A, \mathscr{B}_A)$, where $\mathscr{B}_A$ is the Borel σ-algebra on A. The function f in this case is $f(\alpha) = \alpha_0$, the projection of the word α onto the zeroth coordinate.

3. $T(\alpha) = \alpha + \rho \bmod [0, 2\pi[$, with $\rho/2\pi \in \mathbb{R} \backslash \mathbb{Q}$, an irrational rotation, ergodic with respect to the Lebesgue measure μ_L on $\mathbb{R}$. The function f is chosen as $f(\alpha) = \alpha - \pi$, $\forall \alpha \in [0, 2\pi[$ and $\varepsilon \in \mathbb{Z} \backslash \{0\}$.

4. The transformation is the same as that of Case 3, but the function $f(\alpha)$ is an analytic function in a complex strip $|\mathrm{Im}\, \alpha| < \Delta_\alpha$.

2.1.2. Stochastic Perturbation. We introduce the perturbation of stochastic type by replacing at each step $n \in \mathbb{N}$ the unperturbed frequency ω with $\omega + \varepsilon \alpha_n$, where α_n is the realization of a stochastic process. In particular we choose $\alpha_n(\xi)$, $\xi \in \Omega$, an i.i.d. real random process on some probability space $(\Omega, \mathscr{B}_\Omega, \mu_\Omega)$ and the distribution of μ_Ω is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Note that the Case 2 of the deterministic perturbations is a particular case of the stochastic one with ξ an element of the space A, $\alpha_n(\xi)$ the nth coordinate of ξ, but with an atomic distribution given by the measure (2.1.4). Corresponding to this perturbation, the map at the nth iterate step assumes the form

$$M_p\colon \begin{cases} \theta_{n+1} = \theta_n + \omega + \varepsilon \alpha_n(\xi) & \bmod [0, 2\pi[\\ j_{n+1} = j_n + V(\theta_n), & n \geqslant 0 \end{cases} \tag{2.1.5}$$

We are also interested in the behavior of the mapping M_p for large ε and we will show that for such values the statistical properties of M_p recover those of the standard map in the region of large values of the perturbation parameter, say K. We recall that the evolution of the angle variable in the standard map for large K is supposed to be independent of the action j. This hypothesis, the so-called random phase approximation, can be put on a rigorous basis for the sawtooth map.[7-10] In our model the influence of j on the angle is simulated by the stochastic process $\varepsilon \alpha_n$ and is not trivial in the sense that the random variables entering the definition of the diffusion coefficient (see later) will be, in general, nonindependent, which is precisely what happens for the standard map (we come back to this point in Section 2.2).

2.2. The Diffusion Coefficient

The diffusion coefficient we are going to introduce is defined in terms of the variation of the integral of motion J under perturbation rather than in terms of the variation of the original action j. We first set some useful notations to unify the various modulations quoted in the preceding section. Each perturbation acts on a probability space $\mathscr{P} = (\Omega, \mathscr{B}_\Omega, \mu_\Omega)$ as a stochastic or deterministic process: we denote it by $\alpha_n(\xi)$, $\xi \in \Omega$, $n \geqslant 0$, where ξ must be thought of as the realization of the process in the stochastic case or the initial value in the iteration of the dynamical system in the deterministic case.[5] We then call $\mathscr{T} \equiv (\mathbb{T}^1, \mathscr{B}_L, \mu_L)$ the probability space constituted by the torus $\mathbb{T}^1$ carrying the Borel σ-algebra $\mathscr{B}_L$ and the Lebesgue measure μ_L: it will be the space of the initial angles θ_0. If the perturbation is absent ($\varepsilon = 0$), the value of J given by (2.1.2) is constant under iteration, that is,

$$J_{n+1} \overset{\text{def}}{=} J(\theta_{n+1}, j_{n+1}) \overset{\text{def}}{=} j_{n+1} - \sum_{k=-\infty}^{k=+\infty}{}' \frac{V_k}{e^{ik\omega} - 1} e^{ik\theta_{n+1}} = \text{const} \quad (2.2.1)$$

If we now perturb the frequency according to (2.1.3) or (2.1.5), the left-hand side of (2.2.1) will generally change and this can be geometrically interpreted as a jump between different horizontal curves $J = \text{const}$ on the cylinder parametrized by the coordinates (θ, J), which are the invariant curves of the unperturbed system. A straightforward iteration of Eq. (2.1.2) gives for the variation of J_n in the case (a) the following expression:

$$\begin{aligned} J_{n+1} - J_0 &= \sum_{m=0}^{n} \sum_{k=-\infty}^{k=+\infty}{}' \frac{V_k}{e^{ik\omega} - 1} e^{ik[\theta_0 + (m+1)\omega]} \\ &\quad \times [e^{ik\varepsilon(f(\alpha_{-1}) + f(\alpha_0) + \cdots + f(\alpha_{m-1}))} - e^{ik\varepsilon(f(\alpha_{-1}) + f(\alpha_0) + \cdots + f(\alpha_m))}] \end{aligned} \quad (2.2.2)$$

where $\alpha_{-1} := 0$ and $f(\alpha_{-1}) := 0$ are introduced to simplify future notations. The same expression also holds in the stochastic case (b) by taking the identity as the function f. If we keep J_0 fixed, the process $(J_{n+1} - J_0)$ is well defined on the product space $\mathscr{P} \times \mathscr{T}$; we denote with $\mathbf{E}(\cdot)$ the expectation with respect to the product measure $\mu_\Omega \times \mu_L$. We are now ready to define the diffusion coefficient as the limit, if it exists:

$$D(J_0) = \lim_{n \to +\infty} \frac{\mathbf{E}((J_{n+1} - J_0)^2)}{2(n+1)} \quad (2.2.3)$$

It has to be pointed out that in the case (b) of the stochastic perturbation the analytical estimate of the diffusion coefficient is the same if one considers the true action j:

$$D(j_0) = \lim_{n \to +\infty} \frac{\mathbf{E}((j_{n+1} - j_0)^2)}{2(n+1)} \quad (2.2.4)$$

Nevertheless the choice of J seems to be very natural, especially if one is concerned with not only the diffusion coefficient itself but also the limit dis-

[5] We often write α_n instead of $\alpha_n(\xi)$ when no confusion should arise.

tribution. Numerical simulations show that the distribution of J converges to a Gaussian much faster than the distribution of j, even if the limit variance tends to the same value. The same occurrence is shown by numerical simulations for more complex weakly anisochronous systems.

The main result of this paper is the following:

2.3. Main Result

1. For the mapping $T(\alpha) = 2\alpha \bmod [0, 1[- \frac{1}{2}$, $\alpha \in [-\frac{1}{2}, \frac{1}{2}[$, the diffusion coefficient $D(J_0)$ exists and is independent of J_0 for $\varepsilon \in \mathbb{R}$; in particular, for $\varepsilon \neq 0$ it is given by

$$D(J_0) = \frac{1}{2} \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 + \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 \sum_{r=0}^{+\infty} \cos[k\omega(r+1)] \times \frac{\sin \frac{1}{2} k\varepsilon(1 - 1/2^{r+1})}{\frac{1}{2} k\varepsilon(1 - 1/2^{r+1})} \prod_{j=1}^{r+1} \cos \frac{\varepsilon}{2} k \left(1 - \frac{1}{2^j}\right) \tag{2.3.1}$$

2. For the Bernoulli automorphism $T(\alpha) = \sigma(\alpha)$, with states $s_1,..., s_N$, $s_i \in \mathbb{R}$, $i = 1,..., N$, and nonnegative probabilities $p_1,..., p_N$, we distinguish two cases:

(i) If $\exists \gamma, \mu > 0$ such that $\forall k \in \mathbb{Z} \backslash \{0\}$

$$\left| \sum_{\substack{j,j'=1 \\ j'<j}}^{N} p_j p_{j'} \sin^2 \frac{k\varepsilon}{2} (s_j - s_{j'}) \right|^{-1} \leqslant \gamma \, |k|^{\mu} \tag{2.3.2}$$

the diffusion coefficient $D(J_0)$ exists, is positive, independent of J_0, and given by

$$D(J_0) = \mathrm{Re} \left\{ \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 \left[\frac{1}{2} + \frac{Q_k e^{ik\omega}}{1 - Q_k e^{ik\omega}} \right] \right\} \tag{2.3.3}$$

where we have set $Q_k = \sum_{j=1}^{N} p_j e^{ik\varepsilon s_j}$.

Moreover, denoting by $D(\theta_0, J_0)$ the diffusion coefficient defined by taking the expectation of the process $(J_{n+1} - J_0)^2$ in (2.2.3) with respect to the measure μ_{L} only, we have that $D(\theta_0, J_0) = D(J_0)$, which is therefore independent of θ_0 and J_0.

(ii) If $\exists q_1, q_2,..., q_N, h, l \in \mathbb{Z}$, with $h, l \neq 0$, such that $\varepsilon s_j + \omega = (2\pi/h)\, q_j$, $\forall j = 1, 2,..., N$ and $V_{hl} \neq 0$, then the diffusion coefficient $D(J_0)$ is infinite.

3. When $T(\alpha)=\alpha+\rho \bmod [0, 2\pi[$, $f(\alpha)=\alpha-\pi$ $\forall\alpha\in[0, 2\pi[$, and $\varepsilon\in\mathbb{Z}\setminus\{0\}$, the diffusion coefficient $D(J_0)$ coincides with its quasilinear value:

$$D(J_0)=D_{\mathrm{ql}} \stackrel{\mathrm{def}}{=} \frac{1}{2} \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 \tag{2.3.4}$$

4. For $T(\alpha)=\alpha+\rho \bmod [0, 2\pi[$ and $f(\alpha)=\cos\alpha$, we assume the following hypotheses:

(i) (ω, ρ) satisfy the Brjuno condition[27]

$$\lim_{m\to\infty} \frac{\log \eta_m}{m}=0 \tag{2.3.5}$$

where η_m is defined according to

$$\eta_m=\min_{|k|+|h|\leqslant m} |e^{i(h\omega+k\rho)}-1|, \qquad (h, k)\in\mathbb{Z}^2\setminus\{(0, 0)\} \tag{2.3.6}$$

(ii) The potential $V(\theta)$ is a periodic analytic function in the complex strip $|\mathrm{Im}\,\theta|<\beta$ with vanishing mean and in this domain is bounded by $|V(\theta)|\leqslant C'$.

Then diffusion coefficient is zero for *any* value of ε.

5. In the case of the real random process $\alpha_n(\xi)$, on the probability space $(\Omega, \mathscr{B}_\Omega, \mu_\Omega)$, with the distribution μ_Ω absolutely continuous with respect to the Lebesgue measure, the diffusion coefficient $D(J_0)$ exists, is positive, is independent of J_0, and coincides with $D(\theta_0, J_0)$ for any $\theta_0\in[0, 2\pi[$ and any $\varepsilon\in\mathbb{R}$. The expression turns out to be

$$D(\theta_0, J_0)=D(J_0)=\sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2} \times \mathrm{Re}\left[1-\mathscr{X}(\varepsilon k)-e^{ik\omega}\frac{[1-\mathscr{X}(\varepsilon k)]^2}{1-e^{ik\omega}\mathscr{X}(\varepsilon k)}\right] \tag{2.3.7}$$

where $\mathscr{X}(z)$ is the characteristic function of the probability measure μ_Ω, given by

$$\mathscr{X}(z) \stackrel{\mathrm{def}}{=} \int_\Omega e^{iz\alpha_m(\xi)}\, d\mu_\Omega(\xi), \qquad \forall z\in\mathbb{R} \tag{2.3.8}$$

and independent on $m\in\mathbb{N}$.

In Case 5 the positivity of the diffusion coefficient can be proved in a rigorous way by using (2.3.7) and noting that, under the conditions and with the notations of the Main Result, $\forall \varepsilon \in \mathbb{R}\backslash\{0\}$ and $\forall k \in \mathbb{Z}\backslash\{0\}$ the following equality holds:

$$\operatorname{Re}\left[1-\mathscr{X}(\varepsilon k)-e^{ik\omega}\frac{[1-\mathscr{X}(\varepsilon k)]^2}{1-e^{ik\omega}\mathscr{X}(\varepsilon k)}\right]=2\sin^2\frac{k\omega}{2}\,\frac{1-|\mathscr{X}(\varepsilon k)|^2}{|1-e^{ik\omega}\mathscr{X}(\varepsilon k)|^2} \tag{2.3.9}$$

In an analogous way we prove positivity for the Bernoulli modulation [Case 2(i)], whereas Case 3 is trivial. As for part 1, there is no analytical result, but nevertheless we can give strong numerical evidence for positivity by computing the partial sum of the series:

$$M_1(\varepsilon; k, \omega) \stackrel{\text{def}}{=} \frac{1}{2}+\sum_{r=0}^{+\infty}\cos[k\omega(r+1)]\,\frac{\sin\frac{1}{2}k\varepsilon(1-1/2^{r+1})}{\frac{1}{2}k\varepsilon(1-1/2^{r+1})} \times \prod_{j=1}^{r+1}\cos\frac{\varepsilon}{2}k\left(1-\frac{1}{2^j}\right) \tag{2.3.10}$$

with fixed $k \in \mathbb{Z}\backslash\{0\}$. A simple analytical estimate of the remainder for a given truncation order allows us to state exactly the positivity of the limit (2.3.10) and therefore of the diffusion coefficient $D(J_0)$, provided that the assigned value of k corresponds to a nonvanishing Fourier component of the potential $V(\theta)$ ($V_k = V_{-k} \neq 0$). Two thousand numerical trials have been performed for randomly generated values of the perturbation parameter $\varepsilon \in]0, 100]$ and of the index $k \in]0, 1000]$, by taking as a

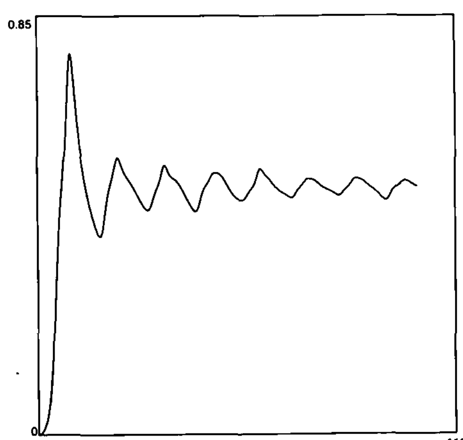

Fig. 1. The limit $M_1(\varepsilon; k, \omega)$ as a function of the perturbation parameter $\varepsilon > 0$ for $k = 1$. For $\varepsilon \gg 0$, the term is very close to $+1/2$, in accordance with the quasilinear estimate.

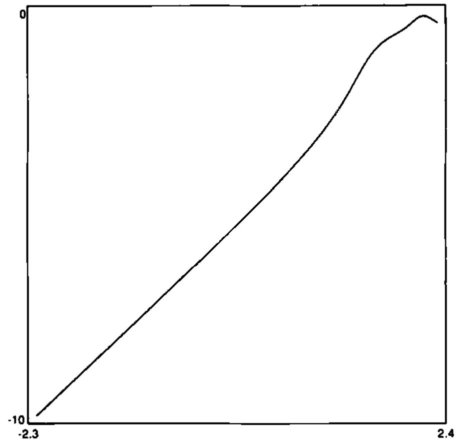

Fig. 2. The limit $M_1(\varepsilon; k, \omega)$ as a function of the perturbation parameter $\varepsilon > 0$ for $k = 1$. The bilogarithmic diagram shows that for $\varepsilon \simeq 0$ the dependence on ε is quadratic.

Diophantine frequency $\omega/2\pi$ the classical golden mean. For most of the trials the above positivity test turns out to be satisfied with a truncation order in (2.3.10) of some hundreds. Only for some particular choices of ε and k (in particular for $\varepsilon \simeq 0$, where the analytical estimate of the remainder does not converge very fast to zero by increasing the order of

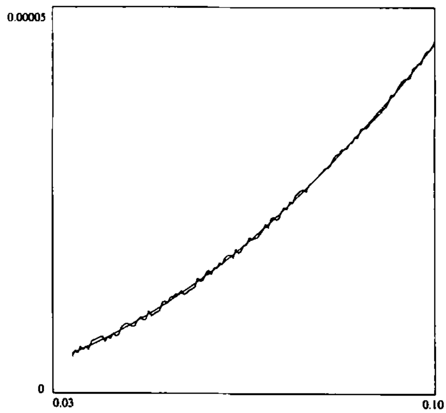

Fig. 3. The limit $M_1(\varepsilon; k, \omega)$ as a function of the parameter $\varepsilon > 0$ for $k = 1$. Smaller values than those in Fig. 2 are considered. Truncation errors in the computation of $M_1(\varepsilon; k, \omega)$ are analytically estimated tobe less than 10^{-6}, in agreement with the parabolic fit. The result confirms the quadratic dependence on ε suggested by Fig. 1.

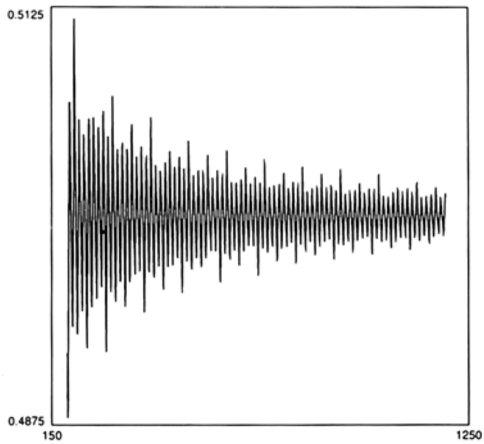

Fig. 4. The limit $M_1(\varepsilon; k, \omega)$ as a function of the parameter $\varepsilon > 0$ for $k = 1$. For $\varepsilon \gg 0$, even if the value is very close to $+1/2$, oscillations of decaying amplitude persist.

truncation) we need to consider a larger number of terms (i.e., $\simeq 10^5$) in order to verify positivity, and no ambiguous case occurs.

By the same method we have obtained the graphs in Figs. 1–4, showing the behavior of the limit $M_1(\varepsilon; k, \omega)$ as a function of the perturbation parameter $\varepsilon > 0$ for $k = 1$. For $\varepsilon \simeq 0$ the dependence on ε is quadratic, whereas in the opposite case $|\varepsilon| \gg 0$ the term $M_1(\varepsilon; k, \omega)$ is very close to $+1/2$, in accordance with the quasilinear estimate.

Even though we do not illustrate the proof here, the latter property can be showed analytically, so that our model satisfies the random phase approximation.

A rigorous proof of the convergence to zero of $M_1(\varepsilon; k, \omega)$ for $\varepsilon \to 0$ can also be given.

2.4. Ergodic Properties

In the case of deterministic processes, the quantity $J_n - J_0$ can be viewed as an observable on the invariant set of the dynamical system F: $\mathbb{T}^1 \times \Omega$ onto itself, defined by

$$F: \quad \begin{cases} \alpha' = T(\alpha), & \alpha \in \Omega \\ \theta' = \theta + \omega + \varepsilon f(\alpha) & \mathrm{mod}[0, 2\pi[\end{cases} \tag{2.4.1}$$

It is easily seen that the dynamical system F is a skew product over T, so that it preserves the product measure $\mu_L \times \mu_\Omega$.[28] This measure is therefore the natural one to take the expectation of the process $(J_n - J_0)^2$. We now set

$$\Delta J_n \stackrel{\text{def}}{=} J_n - J_0 = \sum_{l=0}^{n-1} \eta_l(\theta_0, \alpha) \tag{2.4.2}$$

where η_l is for the deterministic perturbations considered, a process on $\mathbb{T}^1 \times \Omega$ of a very complicated form, but of zero mean with respect to the product measure $\mu_L \times \mu_\Omega$. Moreover, the invariance of this measure makes the process η_l *stationary*. Note that the inner sum in (2.2.2) gives the explicit form of this process, but we will never use this fact in the following. If one proves that $\sum_{j=0}^{\infty} |\mathbf{E}(\eta_0, \eta_j)| < +\infty$, then it is easy to see that the diffusion coefficient exists and can be expressed in the following form (the discrete version of the Green–Kubo formula[29]):

$$D = \mathbf{E}(\eta_0^2) + 2 \sum_{j=1}^{\infty} \mathbf{E}(\eta_0, \eta_j) \tag{2.4.3}$$

The existence and the positivity of D is a necessary condition for establishing stronger statistical properties of the system, for example, the central limit theorem for the stochastic process defined by

$$\frac{\Delta J_n}{(Dn)^{1/2}} = \frac{1}{(Dn)^{1/2}} \sum_{j=0}^{n-1} \eta_j(\theta_0, \alpha), \qquad n \geq 0 \tag{2.4.4}$$

or the Donsker invariance principle,[29] which consists in redefining the process $\Delta J_{[nt]}$, $t \in [0, 1]$, without changing its distribution, on a new probability space together with standard Brownian motion $(W(t), t \geq 0)$ and shows that $\Delta J_{[nt]}/(Dn)^{1/2}$ converges weakly to $W(t)$ when $n \to +\infty$. These questions are numerically investigated in the second part of this work.[26] One also expects the existence and the positivity of the diffusion coefficient for the analytic observable J to be related to the properties of ergodicity, weak mixing, and, in particular, mixing of the measure $\mu_L \times \mu_\Omega$. Such properties can actually be proved whenever $T(\alpha) = 2\alpha \bmod [0, 1[- \frac{1}{2}$, $\alpha \in [-\frac{1}{2}, \frac{1}{2}[$, and only partially when $T(\alpha)$ is the irrational rotation of the circle. However, no one of those properties is sufficient to conclude that the random variables $\eta_j(\theta_0, \alpha)$ are $(\mu_L \times \mu_\Omega)$-independent.

Theorem 1.[30] For the case $T(\alpha) = 2\alpha \bmod [0, 1[- \frac{1}{2}$, $\alpha \in [-\frac{1}{2}, \frac{1}{2}[$, we have that the mapping F is trivially conjugated by means of the affine transformation $(x, y) = (\alpha + 1/2, \theta/2\pi)$ with the map

$$F': \begin{cases} x' = 2x \bmod [0, 1[\\ y' = y + \omega' + \varepsilon' x \bmod [0, 1[\end{cases} \tag{2.4.5}$$

where the parameters ω' and ε' are linear functions of the previous ones $\omega' = (1/2\pi)(\omega - \varepsilon/2)$, $\varepsilon' = \varepsilon/2\pi$. Moreover, the mapping F' satisfies the following properties with respect to the Lebesgue measure on $\mathbb{T}^2$:

(i) F' is mixing iff $\varepsilon' \in \mathbb{R}\backslash\mathbb{Q}$.

(ii) F' is weakly mixing iff it is mixing.

(iii) F' is ergodic and not mixing iff $\varepsilon' \in \mathbb{Q}$ and $\omega' \in \mathbb{R}\backslash\mathbb{Q}$.

(iv) F' is not ergodic iff $\varepsilon' \in \mathbb{Q}$ and $\omega' \in \mathbb{Q}$. ■

Actually we also guess that F' is exact when the mixing property holds.

The proof of Theorem 1 is not immediate; instead what is easy to prove is that F' has positive Kolmogorov entropy, whose value is $\log 2$ (the result follows from a theorem of Abramov and Rokhlin[31,32]).

In the case of the irrational rotation $T(\alpha) = \alpha + \rho \bmod[0, 2\pi[$ the ergodicity of the mapping F easily follows when $\rho \in \mathbb{R}\backslash\mathbb{Q}$ and $\varepsilon \in \mathbb{Z}$ by adapting the proof of ergodicity for skew translations of the torus[28]; the latter result can be slightly generalized, as stated in the following theorem, whose proof follows from an argument due to Anzai.[33]

Theorem 2. For Case 3, $T(\alpha) = \alpha + \rho \bmod[0, 2\pi[$, $f(\alpha) = \alpha - \pi$, the map F can be conjugated, by means of a simple scaling, with the map

$$\tilde{F}: \quad \begin{cases} x' = x + \rho' \bmod[0, 1[\\ y' = y + \omega' + \varepsilon' x \bmod[0, 1[\end{cases} \tag{2.4.6}$$

with parameters $\rho' = \rho/2\pi$, $\omega' = \omega/2\pi - \varepsilon/2$ and $\varepsilon' = \varepsilon$.

We have that:

(i) For $\varepsilon' = 0$ the mapping $\tilde{F}$ is ergodic with respect to the Lebesgue measure on $\mathbb{T}^2 = [0, 1[\times [0, 1[$ iff the frequencies ρ' and ω' are rationally independent.

(ii) For $\varepsilon' \in \mathbb{Q}\backslash\{0\}$, $\tilde{F}$ is ergodic.

(iii) If $\exists \varepsilon' \in \mathbb{R}\backslash\mathbb{Q}$ such that $\tilde{F}$ is nonergodic, then $\forall \bar{\omega} \in \mathbb{R}$ and $\bar{\varepsilon} = a/b + c/d \cdot \varepsilon'$, $a, b, c, d \in \mathbb{Z}\backslash\{0\}$ the mapping

$$\bar{F}: \quad \begin{cases} x' = x + \rho' \bmod[0, 1[\\ y' = y + \bar{\omega} + \bar{\varepsilon} x \bmod[0, 1[\end{cases} \tag{2.4.7}$$

is ergodic.

Moreover, if $\bar{\varepsilon} = c/d \cdot \varepsilon$, $c, d \in \mathbb{Z}\backslash\{0\}$, then $\bar{F}$ is ergodic whenever the frequencies ρ' and $-c\omega + d\bar{\omega}$ are rationally independent. ■

3. PROOF OF THE MAIN RESULT 2.3

3.0. Preliminaries

We first give an expression for the expectation value of the process $(J_{n+1}-J_0)^2$. We use here the notations introduced in Section 2.2 and define the following quantities:

(i) $\forall b \in \{-1, 0, 1, 2,...\}$: $S_b(\xi) = =\sum_{n=-1}^{b} \alpha_n(\xi)$, where we set $\alpha_{-1}=0$.

(ii) $\forall a, b \in \{-1, 0, 1, 2,...\}$:

$$\int_\Omega e^{ik\varepsilon(S_a(\xi)-S_b(\xi))}\, d\mu_\Omega(\xi) \stackrel{\text{def}}{=} \begin{cases} G(b+1, a; k\varepsilon) & \text{if } b<a \\ G(b+1, b; k\varepsilon) \stackrel{\text{def}}{=} 1 & \text{if } b=a \end{cases} \tag{3.0.1}$$

A lengthy calculation shows that

$$\begin{aligned} &\mathbf{E}((J_{n+1}-J_0)^2) \\ &\quad = {\sum_{k=-\infty}^{k=+\infty}}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2}\, 2\,\mathrm{Re}\left\{(n+1) - \sum_{m=0}^{n} G(m, m; k\varepsilon)\right. \\ &\qquad + \sum_{d=1}^{n}\sum_{m'=0}^{n-d} e^{ik\omega d}[G(m', m'+d-1; k\varepsilon) - G(m'+1, m'+d-1; k\varepsilon) \\ &\qquad \left. - G(m', m'+d; k\varepsilon) + G(m'+1, m'+d; k\varepsilon)]\right\} \end{aligned} \tag{3.0.2}$$

which can also be rewritten in the form

$$\begin{aligned} &\mathbf{E}((J_{n+1}-J_0)^2) \\ &\quad = {\sum_{k=-\infty}^{k=+\infty}}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2}\, 2\,\mathrm{Re}\left\{(n+1)[1-G(0,0;k\varepsilon)]\right. \\ &\qquad \left. + \sum_{d=1}^{n} e^{ik\omega d}(n+1-d)[2G(0, d-1; k\varepsilon) - G(0, d-2; k\varepsilon) - G(0, d; k\varepsilon)]\right\} \end{aligned} \tag{3.0.3}$$

The proof is easily achieved by observing that the exponential decay of the Fourier coefficients V_k allows one to apply Lebesgue's dominated convergence theorem to the expectation of the process given by (2.2.2), and using stationarity—which comes from the invariance of the measure μ_Ω in the deterministic case.

3.1. Proof of Part 1

We now specialize to the mapping $\alpha_n(\xi) = \tilde{T}^n(\xi) = 2^n\xi \bmod [0, 1[$, where $\xi \in [0, 1[$. We will show that the mapping $T(\xi) = 2\xi \bmod [0, 1[- \frac{1}{2}$, $\xi \in [-\frac{1}{2}, \frac{1}{2}[$, leads essentially to the same result apart from a phase factor and a correction of the unperturbed rotation number ω. For simplicity of notations, we set $\tilde{T} = T$ and we will return to the true mapping T in the beginning of Appendix A.

The integral in definition (3.0.1) can be performed in an explicit way and reduced to an algebraic expression, according to the result stated in the following Step 1.

Step 1. $\forall m = 0, 1, 2,\ldots$ and $\forall k\varepsilon \in \mathbb{R}$, the following relations hold:

$$G(0, m; k\varepsilon) = \frac{e^{2ik\varepsilon(1-2^{-m-1})} - 1}{ik\varepsilon(2^{m+1} - 1)} \sum_{j=0}^{2^m-1} e^{ik\varepsilon[\delta_m(j) - j/2^m]} \tag{3.1.1}$$

and

$$G(0, m; k\varepsilon) = \frac{\sin \frac{1}{2}k\varepsilon(1 - 2^{-m-1})}{\frac{1}{2}k\varepsilon(1 - 2^{-m-1})} e^{ik\varepsilon[(m+1)/2]} \times \prod_{j=0}^{m+1} \cos\left[\frac{k\varepsilon}{2}\left(1 - \frac{1}{2^j}\right)\right] \tag{3.1.2}$$

where the function $\delta_m(j)$ is defined in terms of the binary representation of the integer $j \in \{0, 1,\ldots, 2^m - 1\}$: $j = \sum_{i=0}^{m-1} a_i 2^i$, $a_i \in \{0, 1\}$, through $\delta_m(j) \stackrel{\text{def}}{=} \sum_{i=0}^{m-1} a_i$. ∎

Proof. We start by observing that

$$G(0, m; k\varepsilon) = \sum_{j=0}^{2^m-1} \int_{j/2^m}^{(j+1)/2^m} e^{ik\varepsilon(\xi + T(\xi) + \cdots + T^m(\xi))}\, d\xi \tag{3.1.3}$$

Then one can show by induction that for each fixed $j \in \{0, 1,\ldots, 2^m - 1\}$ and $\forall \xi \in [j/2^m, (j+1)/2^m[$ it must be that $\forall s = 1, 2,\ldots, m$,

$$T^s(\xi) = 2^s\xi - \sum_{r=0}^{s-1} a_{m+r-s} 2^r \in [0, 1[\tag{3.1.4}$$

Substituting the iterate of ξ just computed in the argument of the exponential in (3.1.3) and rearranging the indices in the sums, we get

$$G(0, m; k\varepsilon) = \sum_{j=0}^{2^m-1} e^{ik\varepsilon(\delta_m(j) - 2j)} \int_{j/2^m}^{(j+1)/2^m} e^{ik\varepsilon(2^{m+1}-1)\xi}\, d\xi \tag{3.1.5}$$

which gives Eq. (3.1.1) after having performed the integrations.

As for Eq. (3.1.2), we can simply deduce it from the previous one by proving $\forall m \in \mathbb{N} \cup \{0\}$ and $\forall k\varepsilon \in \mathbb{R}$ the identity

$$\sum_{j=0}^{2^m - 1} e^{ik\varepsilon[\delta_m(j) - j/2^m]} = \frac{1}{2} \prod_{j=0}^{m} (1 + e^{ik\varepsilon(1 - 1/2^j)}) \tag{3.1.6}$$

The latter relationship easily follows by induction by splitting odd and even terms in the left-hand side of (3.1.6) and using the trivial identities

$$\begin{gathered} \delta_{n+1}(2k') = \delta_{n+1}(k') = \delta_n(k') \\ \delta_{n+1}(2k'+1) = \delta_{n+1}(2k') + 1 = 1 + \delta_{n+1}(k') \\ \forall n \in \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \forall k' = 0, 1,..., 2^n - 1 \end{gathered}$$

Indeed suppose that (3.1.6) holds for $m = n \in \mathbb{N}$ and write

$$\begin{aligned} &\sum_{j=0}^{2^{n+1}-1} \exp\left\{ik\varepsilon\left[\delta_{n+1}(j) - \frac{j}{2^{n+1}}\right]\right\} \\ &\quad = \sum_{j'=0}^{2^n - 1} \exp\left\{ik\varepsilon\left[\delta_{n+1}(2j') - \frac{2j'}{2^{n+1}}\right]\right\} \\ &\qquad + \sum_{j'=0}^{2^n - 1} \exp\left\{ik\varepsilon\left[\delta_{n+1}(2j'+1) - \frac{2j'+1}{2^{n+1}}\right]\right\} \end{aligned} \tag{3.1.7}$$

We have then, for the right-hand side of the previous identity

$$\begin{aligned} &\sum_{j'=0}^{2^n - 1} \exp\left\{ik\varepsilon\left[\delta_n(2j') - \frac{j'}{2^n}\right]\right\} \\ &\quad + \sum_{j'=0}^{2^n - 1} \exp\left\{ik\varepsilon[1 + \delta_n(j')] - \left(\frac{j'}{2^n} + \frac{1}{2^{n+1}}\right)\right\} \\ &= \sum_{j'=0}^{2^n - 1} \exp\left\{ik\varepsilon\left[\delta_n(2j') - \frac{j'}{2^n}\right]\right\} \cdot \left\{1 + \exp\left[ik\varepsilon\left(1 - \frac{1}{2^{n+1}}\right)\right]\right\} \end{aligned} \tag{3.1.8}$$

and finally

$$\begin{aligned} &\frac{1}{2} \prod_{j=0}^{n} \left\{1 + \exp\left[ik\varepsilon\left(1 - \frac{1}{2^j}\right)\right]\right\} \cdot \left\{1 + \exp\left[ik\varepsilon\left(1 - \frac{1}{2^{n+1}}\right)\right]\right\} \\ &\quad = \frac{1}{2} \prod_{j=0}^{n+1} \left\{1 + \exp\left[ik\varepsilon\left(1 - \frac{1}{2^j}\right)\right]\right\} \end{aligned} \tag{3.1.9}$$

As (3.1.6) trivially holds for $m = 0$, the result follows by induction. ■

The previous estimates allow us to give the following representation of the diffusion coefficient.

Step 2. For any $\varepsilon \in \mathbb{R}$ the limit (2.2.3) defining the diffusion coefficient is independent of J_0 and can be written in the form

$$D(J_0) = \lim_{n \to +\infty} \frac{\mathbf{E}((J_{n+1} - J_0)^2)}{2(n+1)} = \frac{1}{2} \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 + \lim_{n \to +\infty} D'_n \quad (3.1.10)$$

where

$$D'_n = \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} 4 \sin\left(\frac{k\omega}{2}\right) \frac{1}{n+1}$$
$$\times \operatorname{Re}\left\{ \sum_{r=0}^{n-1} G(0, r; k\varepsilon)\, e^{ik\omega(r+1)} \left[(n-r) \sin\left(\frac{k\omega}{2}\right) + i \cos\left(\frac{k\omega}{2}\right) \right] \right\} \quad (3.1.11)$$

In particular the diffusion coefficient $D(J_0)$ exists if and only if $\lim_{n \to +\infty} D'_n$ is defined.

Proof. Starting from (3.0.3), by a simple manipulation we find

$$\frac{\mathbf{E}((J_{n+1} - J_0))^2}{2(n+1)}$$
$$= \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \operatorname{Re}\left\{ 1 - \frac{n}{n+1} e^{ik\omega} - \frac{1}{n+1} e^{ik\omega n} G(0, n; k\varepsilon) \right\} + D'_n \quad (3.1.12)$$

Since the coefficients $G(0, r; k\varepsilon)$ are bounded by 1, we get

$$\left| \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \operatorname{Re}\left\{ 1 - \frac{n}{n+1} e^{ik\omega} - \frac{1}{n+1} e^{ik\omega n} G(0, n; k\varepsilon) \right\} \right|$$
$$\leqslant 3 \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \quad (3.1.13)$$

so that the limit for $n \to +\infty$ of the first sum in (3.1.12) exists and equals

$$\tfrac{1}{2} \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2$$

which proves the result. ■

Step 3. We now check the existence of $D' = \lim_{n \to +\infty} D'_n$. Using the expression (3.1.2) for the coefficient $G(0, r; \varepsilon)$ and separating the terms

$(n-r)\sin(k\omega/2)$ and $i\cos(k\omega/2)$ within brackets in (3.1.11), this last limit can be written as

$$D' = \lim_{n\to+\infty} B_1(n) + \lim_{n\to+\infty} B_2(n) \tag{3.1.14}$$

where

$$\begin{aligned} B_1(n) = \mathrm{Re}\Bigg\{ & \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2} 4\sin\left(\frac{k\omega}{2}\right)\cdot i\cos\left(\frac{k\omega}{2}\right) \\ & \times \frac{1}{n+1}\sum_{r=0}^{n-1} \frac{\sin k\varepsilon(1-2^{-r-1})}{k\varepsilon(1-2^{-r-1})} \\ & \times \exp\left[ik\varepsilon\frac{r+1}{2} + ik\omega(r+1)\right]\cdot\prod_{j=0}^{r}\cos\left[\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right] \end{aligned} \tag{3.1.15a}$$

and

$$\begin{aligned} B_2(n) = & \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2} 4\sin^2\left(\frac{k\omega}{2}\right)\cdot\frac{1}{n+1} \\ & \times \mathrm{Re}\Bigg\{\sum_{r=0}^{n-1}(n-r)\frac{\sin k\varepsilon(1-2^{-r-1})}{k\varepsilon(1-2^{-r-1})} \\ & \times \exp\left[ik\varepsilon\frac{r+1}{2} + ik\omega(r+1)\right]\cdot\prod_{j=0}^{r}\cos\left[\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right]\Bigg\} \end{aligned} \tag{3.1.15b}$$

It is easy to show that the limit of $B_1(n)$ always exists and takes the value zero. In fact, the term within the sum over k can be easily bounded by $4(|V_k|^2/|e^{ik\omega}-1|^2)$, so that we can interchange the limit into the sum; we simply have now to discuss the existence of this last limit. To this end we distinguish two cases:

(i) $k\varepsilon \notin 2\pi\mathbb{Z}$. The summand of $B_1(n)$ can be bounded from above by

$$4\frac{|V_k|^2}{|e^{ik\omega}-1|^2}\frac{n}{n+1}\frac{1}{n}\sum_{r=0}^{n-1}\prod_{j=0}^{r}\left|\cos k\frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| \tag{3.1.16}$$

Then we have

$$\lim_{r\to+\infty}\prod_{j=0}^{r}\left|\cos k\frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| = 0 \tag{3.1.17}$$

which, by Cesàro, immediately implies the result.

(ii) $k\varepsilon = 2\pi q$, $q \in \mathbb{Z}$. The summand of $B_1(n)$.can be bounded by

$$4 \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \frac{1}{2\pi q} \frac{n}{n+1} \frac{1}{n} \sum_{r=0}^{n-1} |\sin(\pi q 2^{-r})| \tag{3.1.18}$$

But clearly $\lim_{r \to +\infty} |\sin(\pi q 2^{-r})| = 0$, which allows us to apply the Cesàro argument and gives the result.

As for the limit of $B_2(n)$, we have the following:

Proposition 1. $\forall \varepsilon/2\pi \in \mathbb{R} \backslash \{0\}$:

$$\lim_{n \to +\infty} B_2(n) = \sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2 \sum_{r=0}^{+\infty} \frac{\sin \frac{1}{2} k\varepsilon(1 - 2^{-r-1})}{\frac{1}{2} k\varepsilon(1 - 2^{-r-1})}$$
$$\times \cos k\omega(r+1) \cdot \prod_{j=0}^{r+1} \cos\left[\frac{k\varepsilon}{2}\left(1 - \frac{1}{2^j}\right)\right] \tag{3.1.19}$$

The proof consists in an application of Lebesgue's dominated convergence theorem. The main difficulty is to bound the summand in $B_2(n)$ by a convergent series in $k \in \mathbb{Z}$, uniformly on $n \in \mathbb{N}$; this follows from a suitable resummation. Such a resummation is possible because of the very fast convergence of the cosine argument in the iterated products of (3.1.15a) and (3.1.15b), which allows us to replace those products by powers of $\cos(k\varepsilon/2)$ for large $r \in \mathbb{N}$. The required bound is then obtained by applying Abel's inequality, which leads to the computation of a simple geometric series. In the latter, small denominators are treated by means of the Diophantine condition on $\omega/2\pi$ and using the exponential decay of the Fourier coefficients V_k. The detailed proof is quite lengthy and is deferred to Appendix A.

3.2. Proof of Part 2

With the notations given in the statement of the Main Result in Section 2, we can write the function G defined in (3.0.1) as

$$G(0, r; k\varepsilon) = \int_A e^{ik\varepsilon(f(\alpha) + f(\sigma(\alpha)) + \cdots + f(\sigma^r(\alpha)))} \, d\bar{\mu}(\alpha) \tag{3.2.1}$$

which, for the properties of the measure $\bar{\mu}$, can be easily written as

$$G(0, r; k\varepsilon) = \left[\int_A e^{ik\varepsilon f(\alpha)} \, d\bar{\mu}(\alpha)\right]^{r+1} = \left[\sum_{j=1}^{N} p_j e^{ik\varepsilon s_j}\right]^{r+1} \tag{3.2.2}$$

Setting $Q_k = \sum_{j=1}^{N} p_j e^{ik\varepsilon s_j}$, we find that the limit (2.2.3), if it exists, is equal to

$$\lim_{n\to+\infty} \frac{\mathbf{E}((J_{n+1}-J_0)^2)}{2(n+1)}$$
$$= \lim_{n\to+\infty} \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2} 4\sin\frac{k\omega}{2}$$
$$\times \operatorname{Re}\left\{\frac{1}{n+1}\sum_{r=0}^{n-1} Q_k^{r+1} e^{ik\omega(r+1)}\left[(n-r)\sin\frac{k\omega}{2}+i\cos\frac{k\omega}{2}\right]\right\} \tag{3.2.3}$$

where, of course, $|Q_k| \leqslant 1$.

We recall $R(n)$ the sum in the real part of (3.2.3); it can be rewritten as

$$R(n) = \sum_{r=0}^{n-1} (Q_k e^{ik\omega})^{r+1}\left(n\sin\frac{k\omega}{2}+i\cos\frac{k\omega}{2}\right)$$
$$-\sin\frac{k\omega}{2}\cdot\sum_{r=0}^{n-1}(Q_k e^{ik\omega})^{r+1} r \tag{3.2.4}$$

The sum can be computed explicitly and gives

$$R(n) = \sin\frac{k\omega}{2}\cdot Q_k e^{ik\omega}\frac{1}{1-Q_k e^{ik\omega}} n + \sin\frac{k\omega}{2}\cdot Q_k e^{ik\omega}\frac{(Q_k e^{ik\omega})^{n+1}}{(1-Q_k e^{ik\omega})^2}$$
$$+ i\cos\frac{k\omega}{2}\cdot Q_k e^{ik\omega}\frac{1-(Q_k e^{ik\omega})^n}{1-Q_k e^{ik\omega}}$$
$$-\sin\frac{k\omega}{2}\cdot\frac{(Q_k e^{ik\omega})^2}{(1-Q_k e^{ik\omega})^2} \tag{3.2.5a}$$

whenever $Q_k e^{ik\omega} \neq 1$, and

$$R(n) = \frac{n(n+1)}{2}\sin\frac{k\omega}{2} + in\cos\frac{k\omega}{2} \tag{3.2.5b}$$

in the case $Q_k e^{ik\omega} = 1$.

The computation of the limit (3.2.3) is now quite simple and we defer it to Appendix B.

3.3. Proof of Part 3

The proof of part 3 follows in a straightforward way by observing that, for $T(\alpha)=\alpha+\rho \bmod[0, 2\pi[$, with $\rho/2\pi \in \mathbb{R}\backslash\mathbb{Q}$, $\varepsilon \in \mathbb{Z}\backslash\{0\}$, and $f(\alpha)=\alpha-\pi$, $\forall\alpha\in[0, 2\pi[$, we have

$$G(0, m; k\varepsilon)=\int_{[0, 2\pi[} e^{ik\varepsilon(f(\alpha)+f(T(\alpha))+\cdots+f(T^m(\alpha)))}\, d\mu_{\mathrm{L}}(\alpha)=\delta_{m+1,0} \quad (3.3.1)$$

$\forall m \geqslant 0$, with the definition $G(0, -1; k\varepsilon)=1$. As a consequence, we get

$$\frac{\mathbf{E}((J_{n+1}-J_0)^2)}{2(n+1)}=\sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega}-1|^2}\left(1-\frac{n}{n+1}\cos k\omega\right) \quad (3.3.2)$$

which converges to the quasilinear estimate of the diffusion coefficient as $n \to +\infty$.

3.4. Proof of Part 4

We now prove in detail part 4 of the Main Result.

For simplicity we will give an explicit proof for $f(\alpha)=\cos\alpha$, since the generic case does not present any further difficulty. Let us consider the perturbed map M_p rewritten in the form (2.1.5), for an arbitrary $\varepsilon \in \mathbb{R}\backslash\{0\}$:

$$M_p\colon \quad \begin{cases} \theta_n=\theta_{n-1}+\omega+\varepsilon\alpha_n(\xi) & \bmod[0, 2\pi[\\ I_n=I_{n-1}+V(\theta_n) \end{cases} \quad (3.4.1)$$

where the perturbation is given according to $\alpha_n(\xi)=\cos(\xi+n\rho)$, $\xi\in\mathbb{T}^1$. Assume that (ω, ρ) satisfy conditions (2.3.5), (2.3.6), and that the potential $V(\theta)$ is a periodic analytic function in the complex strip $|\mathrm{Im}\,\theta|<\beta$, bounded by $|V(\theta)|\leqslant C'$ and with vanishing mean.

We want to show that the diffusion coefficient is zero for *any* value of ε, since there are *topological barriers* for the action diffusion in the phase space.

By introducing an auxiliary angle ϕ, we can write the map (3.4.1) in the form

$$\begin{aligned} &\phi'=\phi+\rho \quad \bmod[0, 2\pi[\\ &\theta'=\theta+\omega+\varepsilon\cos\phi \quad \bmod[0, 2\pi[\qquad [(\phi, \theta, I)\in\mathbb{T}^2\times\mathbb{R}] \\ &I'=I+V(\theta) \end{aligned} \quad (3.4.2)$$

and it is straightforward to see that the map (3.4.2) has an integral invariant:

$$d_2=I\,d\theta\wedge d\phi \quad (3.4.3)$$

First we change the angle θ by introducing a new angle Θ according to $\theta = \Theta + \varepsilon u(\phi)$, such that the map (3.4.2) reads

$$\begin{aligned} \phi' &= \phi + \rho \\ \Theta' &= \Theta + \omega \qquad\qquad [(\phi, \Theta, I) \in \mathbb{T}^2 \times \mathbb{R}] \\ I' &= I + V(\Theta + \varepsilon u(\phi)) = I + W(\Theta, \phi) \end{aligned} \tag{3.4.4}$$

The following lemma holds:

Lemma 1. The function $u(\phi)$ is explicitly given by

$$u(\phi) = \frac{1}{2} \frac{\sin(\phi - \rho/2)}{\sin(\rho/2)} \tag{3.4.5}$$

and the new potential $W(\Theta, \phi)$ is a periodic analytic function in the complex strip:

$$|\mathrm{Im}\, \Theta| < \Delta \qquad \text{and} \qquad |\mathrm{Im}\, \phi| < \Delta \tag{3.4.6}$$

with vanishing mean, for a suitable choice of $\Delta > 0$. Moreover, in the latter domain the potential is bounded by $|W(\Theta, \phi)| \leqslant C'$.

Proof. It is straightforward to verify that $u(\phi)$ satisfies the homological equation:

$$u(\phi + \rho) - u(\phi) = \cos \phi \tag{3.4.7}$$

which can be solved since $\rho/2\pi$ is irrational by the hypothesis 1 by providing a Fourier expansion and Eq. (3.4.5) is the unique solution with vanishing mean. By definition the new potential W is analytic in the domain $|\mathrm{Im}\, \Theta + \varepsilon u(\phi)| < \beta$; then by using the explicit form of the solution u we can choose Δ according to the equation

$$\sinh \Delta = \frac{2(\beta - \Delta) \sin(\rho/2)}{\varepsilon} \tag{3.4.8}$$

One can easily see that a nonzero solution Δ exists for any value of ε. Finally we observe that the map (3.4.4) has an integral invariant of the form $D_2 = I\, d\Theta \wedge d\phi$, obtained by transforming (3.4.3). Then if we integrate D_2 on the closed surface $I = \text{const}$ we are lead to

$$\int_{I = \text{const}} I\, d\Theta \wedge d\phi = \int_{I = \text{const}} (I + W(\Theta, \phi))\, d\Theta \wedge d\phi \tag{3.4.9}$$

so that the mean value of the potential W vanishes. ■

Remark. In the case of a general analytic function $f(\alpha)$, we have to consider the whole Fourier expansion of $f(\alpha)$ in the r.h.s. of Eq. (3.4.7); then the existence of an analytic solution is guaranteed by Brjuno's condition (2.3.5).

In order to prove the existence of topological barriers for the action diffusion of the initial map (3.4.1), we shall prove that the phase space of the map (3.4.4) is completely foliated by analytical invariant surfaces; indeed the projections of these surfaces on the initial space (θ, I) provide bounded invariant sets for the dynamics of the map (3.4.1). Let us introduce the new action J according to $I = J + v(\Theta, \phi)$, where $v(\Theta, \phi)$ is a periodic analytic function of vanishing mean, such that J is a first integral of motion of the map (3.4.4). Then it is straightforward to see that v has to satisfy the homological equation

$$v(\Theta + \omega, \phi + \rho) - v(\Theta, \phi) = W(\Theta, \phi) \tag{3.4.10}$$

so that, by expanding in the Fourier basis, the solution v can be formally written according to

$$v(\Theta, \phi) = \sum_{(h,k) \neq (0,0)} \frac{W_{h,k}}{e^{i(h\omega + k\rho)} - 1} e^{i(h\Theta + k\phi)} \tag{3.4.11}$$

Since the potential $W(\Theta, \phi)$ is analytic in the strip (3.4.6), we can estimate the Fourier coefficients according to $|W_{h,k}| \leqslant Ce^{-(|h| + |k|)\Delta}$. The series (3.4.11) will be absolutely convergent in the strip (3.4.6) if the series

$$\sum_{m \geqslant 1} \frac{e^{-m\Delta}(m+1)}{\eta_m} \tag{3.4.12}$$

is convergent. By using the Hadamard criterion, we obtain the condition

$$\lim_{m \to \infty} \left(\frac{e^{-m\Delta}(m+1)}{\eta_m} \right)^{1/m} < 1 \tag{3.4.13}$$

but by using the Brjuno condition (2.3.5) we have $\lim_{m \to \infty} (\eta_m)^{-1/m} = 1$, so that Eq. (3.4.13) is always satisfied since $e^{\Delta} > 1$ for any $\Delta > 0$. This completes the proof of the Main Result, part 4. ∎

Remark. An analogous proof for the existence of topological barriers to the diffusion can be given for any analytic quasiperiodic modulation $f(\alpha_1, \ldots, \alpha_d)$ assuming a Brjuno condition for the frequencies $(\omega, \rho_1, \ldots, \rho_d)$.

3.5. Proof of Part 5

We start by considering the variance of the process $J_{n+1} - J_0$ given by (3.0.2) in Section 3. We first recall some notations:

$$S_b(\xi) \stackrel{\text{def}}{=} \sum_{n=-1}^{b} \alpha_n(\xi) \qquad \forall b \in \{-1, 0, \ldots\}, \quad \alpha_{-1} = 0 \tag{3.5.1}$$

where $\alpha_n(\xi)$, $\xi \in \Omega$, is an i.i.d. real random process on some probability space $(\Omega, \mathscr{B}_\Omega, \mu_\Omega)$ and the distribution of μ_Ω is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. The expectation $\mathbf{E}$ is taken with respect to μ_Ω, and by $\mathscr{X}(\varepsilon k)$ we denote the characteristic function

$$\mathscr{X}(z) \stackrel{\text{def}}{=} \int_\Omega e^{iz\alpha_n(\xi)}\, d\mu_\Omega(\xi)$$

which is independent of n since the process is identically distributed. For any $a > b$ the function G now takes the form

$$G(b+1, a; ke) \stackrel{\text{def}}{=} \mathbf{E}[e^{ik\varepsilon(S_a(\xi) - S_b(\xi))}] = \mathscr{X}(\varepsilon k)^{a-b} \tag{3.5.2}$$

A straightforward manipulation allows to rewrite the expression (3.0.2) as

$$\begin{aligned}
&\frac{\mathbf{E}((J_{n+1} - J_0)^2)}{2(n+1)} \\
&= \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \frac{1}{2(n+1)} \Big\{ 2(n+1)[1 - \operatorname{Re} \mathscr{X}(\varepsilon k)] \\
&\quad + 2e^{ik\omega}[2\mathscr{X}(\varepsilon k) - \mathscr{X}(\varepsilon k)^2 - 1] \frac{n - (n+1)\, e^{ik\omega}\mathscr{X}(\varepsilon k) + [e^{ik\omega}\mathscr{X}(\varepsilon k)]^{n+1}}{[e^{ik\omega}\mathscr{X}(\varepsilon k) - 1]^2} \Big\}
\end{aligned} \tag{3.5.3}$$

For $\varepsilon \neq 0$ ($\varepsilon = 0$ being trivial) the limit for $n \to +\infty$ of the argument in the sum (3.5.3), which we call $P(n)$, exists and is equal to

$$\lim_{n \to +\infty} P(n) = \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \operatorname{Re} \left\{ 1 - \mathscr{X}(\varepsilon k) + \frac{e^{ik\omega}[2\mathscr{X}(\varepsilon k) - \mathscr{X}(\varepsilon k)^2 - 1]}{1 - e^{ik\omega}\mathscr{X}(\varepsilon k)} \right\} \tag{3.5.4}$$

as, of course, $\mathscr{X}(\varepsilon k)\, e^{ik\omega} \neq 1$, $\forall k \in \mathbb{Z} \backslash \{0\}$. Moreover, it is easy to prove that

$$|P(n)| \leqslant \frac{|V_k|^2}{|e^{ikk} - 1|^2} \left[2 + 8 \frac{1}{[1 - |\mathscr{X}(\varepsilon k)|]^2} \right] \tag{3.5.5}$$

We show that the previous bound (3.5.5) is summable in $k \in \mathbb{Z}\backslash\{0\}$. To this end, notice that $\forall z \in \mathbb{R}$ the characteristic function $\mathscr{X}(z)$ is the Fourier transform of a density $\rho \in L^1(\mathbb{R})$; therefore, by the Riemann–Lebesgue theorem it is continuous and $\lim_{z \to +\infty} \mathscr{X}(z) = \lim_{z \to -\infty} \mathscr{X}(z) = 0$. As a consequence there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall k \in \mathbb{Z}\backslash\{0\}$, $|k| \geqslant n_\varepsilon$, one has $|\mathscr{X}(\varepsilon k)| \leqslant \frac{1}{2}$ and $1/\{1 - |\mathscr{X}(\varepsilon k)|\} \leqslant 2$. Furthermore (see Lukacs,[34] p. 16) since the distribution of the density ρ

$$F(x) \overset{\text{def}}{=} \int_{[-\infty, x]} \rho(\xi) \, d\xi \tag{3.5.6}$$

is continuous in $\mathbb{R}$ (by Lebesgue), it cannot be of lattice type (such a distribution being characterized by equally spaced discontinuities in $\mathbb{R}$). Then $|\mathscr{X}(z)| < 1$, $\forall z \in \mathbb{R}\backslash\{0\}$, and for $k \in \mathbb{Z}\backslash\{0\}$, $|k| < n_\varepsilon$, we can write

$$|\mathscr{X}(\varepsilon k)| \leqslant \sup_{z \in [|\varepsilon|, n_\varepsilon |\varepsilon|] \cup [-n_\varepsilon |\varepsilon|, -|\varepsilon|]} |\mathscr{X}(z)| \overset{\text{def}}{=} \rho_\varepsilon < 1 \tag{3.5.7}$$

i.e., $1/\{1 - |\mathscr{X}(\varepsilon k)|\} \leqslant 1/(1 - \rho_\varepsilon)$. Denoting by M_ε the finite expression $\operatorname{Sup}\{1/(1 - \rho_\varepsilon; 2\}$, we obtain for the upper bound (3.5.5)

$$|P(n)| \leqslant \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} 2(1 + 4M_\varepsilon^2) \tag{3.5.8}$$

clearly summable in $k \in \mathbb{Z}\backslash\{0\}$ because of the exponential decay of $|V_k|$ and the Diophantine hypothesis on $\omega/2\pi$. We can finally apply the Lebesgue theorem and get

$$D(J_0) = \operatorname{Re} \sum_{k=-\infty}^{k=+\infty}{}' \frac{|V_k|^2}{|e^{ik\omega} - 1|^2} \left\{ 1 - \mathscr{X}(\varepsilon k) + \frac{e^{ik\omega}[2\mathscr{X}(\varepsilon k) - \mathscr{X}(\varepsilon k)^2 - 1]}{1 - e^{ik\omega}\mathscr{X}(\varepsilon k)} \right\} \tag{3.5.9}$$

which is obviously independent of J_0, so that we call it simply D in the following.

4. COMMENTS AND CONCLUSIONS

The graphs in Figs. 1–4 illustrate the typical behavior of the diffusion coefficient as a function of the coupling parameter ε in the case of the Markov modulation. Analogous computations can be performed for all the other classes of noise and lead essentially to the same portrait: the diffusion coefficient is a continuous function of ε, vanishes at $\varepsilon = 0$, and tends to its quasilinear estimate D_{ql} in the limit $\varepsilon \to +\infty$. In an intermediate range of

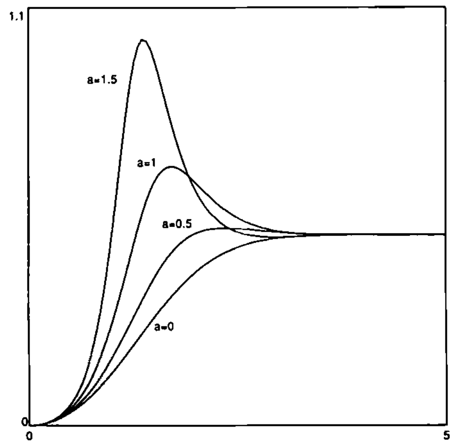

Fig. 5. The same as in Fig. 1, for the case of a Gauss distribution with variance 1 and various choices of the mean a.

ε the coefficient shows oscillations and can be quite far from the corresponding quasilinear value. Even if the previous trend seems to be quite general, it is also possible to show some situations where a different behavior occurs (Figs. 5 and 6). In particular, we have already mentioned in Section 3.2, case (ii), that for a Bernoulli noise there exist choices of the

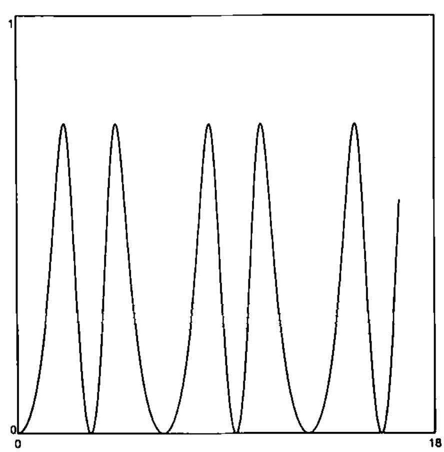

Fig. 6. The same as in Fig. 1, for a Bernoulli modulation with $s_1 = -1$, $s_2 = +1$, and statistical weights $p_1 = p_2 = 1/2$. The oscillating regime indicates that the random phase approximation fails.

parameters of the map for which the diffusion coefficient is singular; these singularities admit an interesting physical interpretation, as a linear variation in time of the action variable (ballistic motion).[22,35,23] Anyway, according to Proposition B1 of Appendix B, the hypothesis (i) holds with probability one for a random choice of parameters, so that item (ii) actually describes a completely exceptional situation. It is also to be pointed out (see Fig. 6) that for any choice of the parameters the model with the Bernoulli modulation violates the random phase approximation in the sense that for large values of the perturbation parameter ε the diffusion coefficient does not tend to its quasilinear part. This is evident by looking at formula (2.3.3): in fact, due to the structure of the coefficient Q_k, the limit of $D(J_0)$ does not exist when $\varepsilon \to +\infty$. Finally, we want to emphasize that with both Bernoulli and stochastic modulations the same expressions for the diffusion coefficient—(2.3.3) and (2.3.7), respectively—hold even if we do not average with respect to the angle variable and take any value for the initial angle (see ref. 35 for a complete proof).

For the sake of completeness we specialize the result on the stochastic noise to the Gauss and the uniform distribution, respectively. In the case of a Gauss distribution with mean a and standard deviation $\sigma > 0$ the diffusion coefficient is given by (3.5.9) with the characteristic function

$$\mathscr{X}(\varepsilon k) = \exp\left(iak\varepsilon - \frac{\sigma^2}{2} k^2 \varepsilon^2\right) \tag{4.1}$$

For the uniform distribution the density is

$$\rho(\alpha) \stackrel{\text{def}}{=} \begin{cases} 1/(2r) & \text{if} \quad |\alpha - a| \leqslant r \\ 0 & \text{if} \quad |\alpha - a| > r \end{cases} \tag{4.2}$$

and the characteristic function reads

$$\mathscr{X}(\varepsilon k) = e^{ik\varepsilon a} \frac{\sin k\varepsilon r}{k\varepsilon r} \tag{4.3}$$

Figure 5 shows that for the Gauss distribution the diffusion coefficient converges to the quasilinear estimate as $\varepsilon \to +\infty$, but in a different way according to the mean value a of the distribution itself. In particular the convergence is monotonic for $a = 0$, whereas when $a \neq 0$ an oscillating behavior arises with regions of superlinear regime. As outlined in Section 2.5, we remark that existence and positivity of the diffusion coefficient represent a first step in the investigation of stronger statistical properties of the model—central limit theorem, Donsker's invariance principle. A numerical analysis of these properties is the subject of a further paper.[26]

APPENDIX A. PROOF OF PROPOSITION 1

Before giving the proof of the existence of the limit of $B_2(n)$, we must return to the true map $T(\xi)=2\xi \bmod [0,1[-\frac{1}{2}$, $\xi\in[-\frac{1}{2},\frac{1}{2}[$. It is easy to see that the integrals (3.1.3) defining the function $G(0,m;k\varepsilon)$ computed with the true map are given by

$$\int_{-1/2}^{1/2} e^{ik\varepsilon(\xi+T(\xi)+\cdots+T^m(\xi))}\,d\mu_\Omega(\xi)$$
$$=\frac{\sin\frac{1}{2}k\varepsilon(1-2^{-m-1})}{\frac{1}{2}k\varepsilon(1-2^{-m-1})}\prod_{j=0}^{m+1}\cos\left[\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right] \tag{A.1}$$

The similarity of the new expression of $G(0,m;k\varepsilon)$ with respect to the previous one is due to the fact that the dynamical systems $\tilde{T}$ on $[0,1[$ and T on $[-\frac{1}{2},\frac{1}{2}[$ are trivially conjugated by a simple translation on $\mathbb{R}$.

Consequently, $B_2(n)$ now takes the form

$$B_2(n)=\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2\frac{1}{n+1}\sum_{r=0}^{n-1}(n-r)\frac{\sin\frac{1}{2}k\varepsilon(1-2^{-r-1})}{\frac{1}{2}k\varepsilon(1-2^{-r-1})}$$
$$\times\cos[k\omega(r+1)]\cdot\prod_{j=0}^{r+1}\cos\left[\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right] \tag{A.2}$$

whereas $B_1(n)$ is again zero and the quasilinear estimate D_{ql} of the diffusion coefficient does not change. From now on we refer to $B_2(n)$ as given by (A.2).

To check the existence of the limit of $B_2(n)$ we will distinguish two cases:

(i) $\varepsilon/2\pi\in\mathbb{R}\backslash\mathbb{Q}$.

(ii) $\varepsilon/2\pi\in\mathbb{Q}$.

Case (i). We now consider the case of $\varepsilon/2\pi\in\mathbb{R}\backslash\mathbb{Q}$. The proof consists in an application of Lebesgue's dominated convergence theorem. The main difficulty is to bound the summand in (A.2) by a convergent series in $k\in\mathbb{Z}$, uniformly on $n\in\mathbb{N}$; a nontrivial resummation allows to get such a bound by using the exponential decay of the Fourier coefficients and the Diophantine hypothesis on the rotation number $\omega/2\pi$:

$$
\begin{aligned}
|B_2(n)| \leqslant{} & |V_k|^2 \frac{1}{n+1} \sum_{r=0}^{n-1} (n-r) \left| \frac{\sin \frac{1}{2}k\varepsilon(1-2^{-r-1})}{\frac{1}{2}k\varepsilon(1-2^{-r-1})} - \frac{\sin \frac{1}{2}k\varepsilon}{\frac{1}{2}k\varepsilon} \right| \\
& \times \prod_{j=0}^{r+1} \left| \cos k \frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \right| + |V_k|^2 \frac{1}{n+1} \\
& \times \left| \sum_{r=0}^{n-1} (n-r) \cos k\omega(r+1) \cdot \prod_{j=0}^{r+1} \cos \frac{\varepsilon}{2} k \left(1-\frac{1}{2^j}\right) \right| \\
& \times \left| \frac{\sin \frac{1}{2}k\varepsilon}{\frac{1}{2}k\varepsilon} \right|
\end{aligned}
\tag{A.3}
$$

By applying the mean value theorem to the difference in the absolute value in the first sum, we get a bound of type $Mk\varepsilon 2^{-r-2}$. Consequently the first sum in (A.3) admits the upper bound $M\varepsilon k\, |V_k|^2/2$, which is uniform in $n \in \mathbb{N}$ and integrable in $k \in \mathbb{Z}\backslash\{0\}$. Omitting the factor $|V_k|^2$, we find that the second sum in (A.3), which we call $B_3(n)$, is bounded by

$$
\begin{aligned}
B_3(n) \leqslant{} & \frac{1}{n+1} \left| \sum_{r=0}^{n-1} (n-r) \cos[k\omega(r+1)] \cdot \prod_{j=1}^{r+1} \cos \frac{\varepsilon}{2} k \left(1-\frac{1}{2^j}\right) \right| \\
& \times \left| \frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k} \right|
\end{aligned}
\tag{A.4}
$$

Set

$$
\left\{\frac{k\varepsilon}{2}\right\} \stackrel{\text{def}}{=} \frac{k\varepsilon}{2} \quad \operatorname{mod}[0, \pi[\tag{A.5}
$$

and introduce the distance from the set $\pi\mathbb{Z}$ given by

$$
\operatorname{dist}\left(\frac{k\varepsilon}{2}; \pi\mathbb{Z}\right) \stackrel{\text{def}}{=} \operatorname{Inf}\left\{\left\{\frac{k\varepsilon}{2}\right\}, \pi - \left\{\frac{k\varepsilon}{2}\right\}\right\} \tag{A.6}
$$

which is surely positive $\forall k \in \mathbb{N}$.

We look for an index $\bar{j}(k) \in \mathbb{N}$ such that $\forall_j \in \mathbb{N}$, $j > \bar{j}(k)$,

$$
\frac{1}{2}\frac{k\varepsilon}{2^j} < \frac{1}{2} \operatorname{dist}\left(\frac{k\varepsilon}{2}; \pi\mathbb{Z}\right) \tag{A.7}
$$

A simple calculation shows that a possible choice is given by

$$
\bar{j}(k) \stackrel{\text{def}}{=} 1 + \frac{1}{\log 2} \log\left[\frac{k\varepsilon/2}{\operatorname{dist}(k\varepsilon/2; \pi\mathbb{Z})}\right] \tag{A.8}
$$

Therefore $\forall j \in \mathbb{N}$, $j > \bar{j}(k)$, we have

$$\operatorname{dist}\left(\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right); \pi\mathbb{Z}\right) \geqslant \frac{1}{2}\operatorname{dist}\left(\frac{k\varepsilon}{2}; \pi\mathbb{Z}\right) \tag{A.9}$$

Splitting the sum (A.4) over the intervals $[0, \bar{j}(k)]$ and $[\bar{j}(k)+1, n-1]$, we can bound (A.4) as

$$B_3(n) \leqslant \left|\frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k}\right| [\bar{j}(k)+1] + \left|\frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k}\right|$$
$$\times \left|\sum_{r=\bar{j}(k)+1}^{n-1} \frac{n-r}{n+1} \cos[k\omega(r+1)] \cdot \prod_{j=\bar{j}(k)+1}^{r+1} \cos k\frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| \tag{A.10}$$

By the very definition of $\bar{j}(k)$ we have that $\forall \varepsilon/2\pi$ irrational and $\forall j \in \mathbb{N}$, $j > \bar{j}(k)$,

$$\operatorname{sgn}\left[\cos k\frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right] = \operatorname{sgn}\left(\cos k\frac{\varepsilon}{2}\right) \tag{A.11}$$

We now introduce the following notation $\forall r \in \mathbb{N}$, $\bar{j}(k)+1 \leqslant r \leqslant n-1$:

$$f_r \stackrel{\text{def}}{=} \prod_{j=\bar{j}(k)+1}^{r+1} \left|\cos k\frac{\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| \tag{A.12}$$

and

$$a_r \stackrel{\text{def}}{=} \frac{n-r}{n+1} \cos[k\omega(r+1)] \cdot \left[\operatorname{sgn} \cos\frac{\varepsilon}{2}k\right]^{r+1} \tag{A.13}$$

Since $\forall r = \bar{j}(k)+1,..., n-2$ we clearly have $0 < f_{r+1} \leqslant f_r$, by applying Abel's inequality,[36] we find that the sum in (A.10) admits the bound

$$\left|\sum_{r=\bar{j}(k)+1}^{n-1} a_r \cdot f_r\right| \leqslant A \cdot f_{\bar{j}(k)+1} \tag{A.14}$$

where

$$A = \sup_{s \in \{\bar{j}(k)+1,..., n-1\}} |a_{\bar{j}(k)+1} + \cdots + a_s|$$

Since $f_{\bar{j}(k)+1} \leqslant 1$, we have

$$\left|\sum_{r=\bar{j}(k)+1}^{n-1} a_r \cdot f_r\right| \leqslant A$$

and, moreover, $\forall s = \bar{j}(k)+1,\dots, n-1$,

$$\sum_{r=\bar{j}(k)+1}^{s} a_r \stackrel{\text{def}}{=} \operatorname{Re}\left\{ \sum_{r=\bar{j}(k)+1}^{s} \frac{n-r}{n+1} \left[e^{ik\omega} \left(\operatorname{sgn} \cos \frac{\varepsilon}{2} k \right) \right]^{r+1} \right\}$$
$$= \operatorname{Re}\left\{ \left[e^{ik\omega} \left(\operatorname{sgn} \cos \frac{\varepsilon}{2} k \right) \right]^{n} \frac{1}{n+1} \right.$$
$$\left. \times \frac{d}{dx} \left[\sum_{r=\bar{j}(k)+1}^{s} x^{n-r} \right] \Bigg|_{x = e^{-ik\omega}[\operatorname{sgn} \cos(\varepsilon k/2)]} \right\} \tag{A.15}$$

A standard derivation of the geometric series allows us to bound immediately (A.15) and consequently (A.4) as

$$B_3(n) \leqslant \left| \frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k} \right| [\bar{j}(k)+1] + 4 \left| \frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k} \right|$$
$$\times \left| 1 - e^{-ik\omega} \left(\operatorname{sgn} \cos \frac{\varepsilon}{2} k \right) \right|^{-2} \tag{A.16}$$

Note that in the case $n \leqslant \bar{j}(k)$ the bound is simply given by the first term is (A.16). We have thus proved that the bound is uniform in $n \in \mathbb{N}$. Concerning the summability in $k \in \mathbb{Z}\backslash\{0\}$, we first observe that thanks to the further factor $|V_k|^2$, the bound

$$\left| \frac{\sin \frac{1}{2}\varepsilon k}{\frac{1}{2}\varepsilon k} \right| [\bar{j}(k)+1]$$
$$\leqslant \frac{4}{k\varepsilon} + \frac{1}{\log 2} \log \left(\frac{k\varepsilon/2}{\operatorname{dist}(k\varepsilon/2; \pi\mathbb{Z})} \right)$$
$$\times \left| \operatorname{dist}\left(\frac{k\varepsilon}{2}; \pi\mathbb{Z} \right) \right| \cdot \left| \frac{\sin \operatorname{dist}(k\varepsilon/2; \pi\mathbb{Z})}{\operatorname{dist}(k\varepsilon/2; \pi\mathbb{Z})} \right| \cdot \left| \frac{1}{k\varepsilon/2} \right| \tag{A.17}$$

enables us to apply Lebesgue's dominated convergence theorem. For the term in ω we have

$$\left| 1 - e^{-ik\omega} \left(\operatorname{sgn} \cos \frac{\varepsilon}{2} k \right) \right|^{2} = \begin{cases} |1 - e^{-ik\omega}|^{-2} & \text{if } \operatorname{sgn} \cos \frac{1}{2}\varepsilon k = 1 \\ |1 + e^{-ik\omega}|^{-2} & \text{if } \operatorname{sgn} \cos \frac{1}{2}\varepsilon k = -1 \end{cases} \tag{A.18}$$

After having set $\{x\} \stackrel{\text{def}}{=} x \bmod [-\pi/2, \pi/2[$, we have

$$|e^{ik\omega} - 1|^{-2} = \frac{1}{4 \sin^2(k\omega/2)} \leqslant \frac{\pi^2}{16} \frac{1}{\{k\omega/2\}^2} \tag{A.19}$$

But for a suitable $m \in \mathbb{Z}$ (depending on k), one has

$$\left\{\frac{k\omega}{2}\right\}^{-2} = \left|\frac{k\omega}{2} - \pi m\right|^{-2} = \frac{1}{\pi^2 k^2}\left|\frac{\omega}{2\pi} - \frac{m}{k}\right|^{-2}$$
$$\leqslant \frac{1}{\pi^2}\gamma_\omega^2 k^{2(\mu_\omega - 1)} \tag{A.20}$$

where $\gamma_\omega > 0$ and $\mu_\omega \geqslant 2$ are the parameters defining the Diophantine character of the frequency $\omega/2\pi$.

Hence

$$|e^{ik\omega} - 1|^{-2} \leqslant \tfrac{1}{16}\gamma_\omega^2 k^{2(\mu_\omega - 1)} \tag{A.21}$$

In a similar way we get

$$|e^{ik\omega} + 1|^{-2} \leqslant 4^{\mu_\omega - 2}\gamma_\omega^2 k^{2(\mu_\omega - 1)}$$

and finally, since $\mu_\omega \geqslant 2$,

$$\left|1 - e^{-ik\omega}\left(\operatorname{sgn}\cos\frac{\varepsilon}{2}k\right)\right|^{-2} \leqslant 4^{\mu_\omega - 2}\gamma_\omega^2 k^{2(\mu_\omega - 1)} \tag{A.22}$$

which is surely summable when multiplied by $|V_k|^2\, |(\sin\frac{1}{2}\varepsilon k)/(\frac{1}{2}\varepsilon k)|$, which concludes the proof of case (i).

Case (ii). Let $\varepsilon/2\pi \in \mathbb{Q}$, $\varepsilon = p/q$, $p \in \mathbb{Z}\backslash\{0\}$, and $q \in \mathbb{N}$. The crucial point is again the estimate of (A.2), which, after the position $k = lq + m$, with $l \in \mathbb{Z}$ and $m \in \{0, 1, 2,..., q-1\}$, can be written as

$$\sum_{l=-\infty}^{+\infty}\sum_{m=0}^{q-1}|V_{lq+m}|^2\frac{1}{n+1}\sum_{r=0}^{n-1}(n-r)\prod_{j=0}^{r+1}\cos\left[(lq+m)\pi\frac{p}{q}\left(1 - \frac{1}{2^j}\right)\right]$$
$$\times\cos[(r+1)(lq+m)\omega]\,\frac{\sin[\pi(p/q)(lq+m)(1-2^{-r-1})]}{\pi(p/q)(lq+m)(1-2^{-r-1})} \tag{A.23}$$

We now proceed to an estimation, uniform in $n \in \mathbb{N}$ and summable in $l \in \mathbb{Z}$, of each of the terms in the sum $\sum_{l\in\mathbb{Z}}$, which we simply call $\psi_l(n)$.

In this last term we separate the contributions for $m = 0$ and $m > 0$ in the corresponding sum, thus obtaining

$$\psi_l(n) \leqslant \psi_{l,0}(n) + \psi_{l,1}(n) \tag{A.24}$$

The term $\psi_{l,0}(n)$ is easily seen to be bounded for above by $2\,|V_{lq}|^2$, while

$$\psi_{l,1}(n) \leqslant \sum_{m=1}^{q-1}|V_{lq+m}|^2\sum_{r=0}^{\infty}\prod_{j=0}^{r+1}\left|\cos\left[\pi m\frac{p}{q} - \pi p\left(l + \frac{m}{q}\right)\frac{1}{2^j}\right]\right| \tag{A.25}$$

Notice that $\forall m = 1, 2, \ldots, q-1$ and $\forall l \in \mathbb{Z}$, $\exists \bar{j}(m; l) \in \mathbb{N} / \forall j > \bar{j}(m; l)$:

$$\frac{1}{2^j} \left| \pi p \left(l + \frac{m}{q} \right) \right| < \frac{1}{2} \left| \operatorname{dist} \left(\pi m \frac{p}{q} ; \pi \mathbb{Z} \right) \right| \tag{A.26}$$

It is then possible to choose a suitable $\bar{j}(m; l)$ for which $\forall l \in \mathbb{Z}$, $\forall j > \bar{j}(m; l)$, and independently on $m = 1, 2, \ldots, q-1$,

$$\begin{gathered} \left| \cos \left[\pi m \frac{p}{q} - \pi p \left(l + \frac{m}{q} \right) \frac{1}{2^j} \right] \right| \leqslant 1 \\ \operatorname*{Sup}_{m = 1, \ldots, q-1} \bar{j}(m; l) \leqslant 1 + \frac{1}{\log 2} \log[\pi \, |p| \, (|l| + 1)] \stackrel{\text{def}}{=} \bar{j}(l) < +\infty \end{gathered} \tag{A.27}$$

and

$$\begin{aligned} S \stackrel{\text{def}}{=} \operatorname*{Sup}_{m = 1, \ldots, q-1} \operatorname{Sup} \Bigg\{ & \left| \cos \left[\pi m \frac{p}{q} + \frac{1}{2} \operatorname{dist} \left(\pi m \frac{p}{q} ; \pi \mathbb{Z} \right) \right] \right| ; \\ & \left| \cos \left[\pi m \frac{p}{q} - \frac{1}{2} \operatorname{dist} \left(\pi m \frac{p}{q} ; \pi \mathbb{Z} \right) \right] \right| \Bigg\} < 1 \end{aligned} \tag{A.28}$$

and independent of $l \in \mathbb{Z}$. Using the previous inequalities, we can immediately bound $\psi_{l,1}(n)$ as

$$\psi_{l,1}(n) \leqslant \sum_{m=1}^{q-1} |V_{lq+m}|^2 \left[\bar{j}(l) + 1 + S^2 \frac{1}{1-S} \right] \tag{A.29}$$

which allows us to bound $\psi_l(n)$ as

$$\begin{aligned} |\psi_l(n)| \leqslant 2 \, |V_{ql}|^2 + & \left[1 + \frac{S^2}{1-S} + 1 + \frac{\log \pi \, |p|}{\log 2} + \frac{1}{\log 2} \log(|l| + 1) \right] \\ & \times \sum_{m=1}^{q-1} |V_{lq+m}|^2 \end{aligned} \tag{A.30}$$

clearly summable in $l \in \mathbb{Z}$ because of the exponential decay of $|V_{lq+m}|^2$. We have now to check the existence of the limit $n \to \infty$ of $\psi_l(n)$. As in the irrational case, we define the two quantities

$$\begin{aligned} \varphi_1(n) = \sum_{r=0}^{n-1} \prod_{j=0}^{r+1} & \cos \left[(lq + m) \pi \frac{p}{q} \left(1 - \frac{1}{2^j} \right) \right] \\ & \times \frac{\sin[\pi (p/q)(lq + m)(1 - 2^{-r-1})]}{\pi (p/q)(lq + m)(1 - 2^{-r-1})} \cos[(r+1)(lq + m)\omega] \end{aligned}$$

$$\varphi_2(n)=\frac{1}{n+1}\sum_{r=0}^{n-1} r\prod_{j=0}^{r+1}\cos\left[(lq+m)\pi\frac{p}{q}\left(1-\frac{1}{2^j}\right)\right]$$
$$\times\frac{\sin[\pi(p/q)(lq+m)(1-2^{-r-1})]}{\pi(p/q)(lq+m)(1-2^{-r-1})}\cos[(r+1)(lq+m)\omega] \tag{A.31}$$

If the limits $n\to+\infty$ of the above expressions exist, then $\lim_{n\to+\infty}\psi_l(n)$ will exist in turn and take the form

$$\lim_{n\to+\infty}\psi_l(n)=\sum_{m=0}^{q-1}|V_{lq+m}|^2\,[\lim_{n\to+\infty}\varphi_1(n)-\lim_{n\to+\infty}\varphi_2(n)] \tag{A.32}$$

$\lim_{n\to+\infty}\varphi_1(n)$ trivially exists for $m=1,2,\ldots,q-1$, since the series is absolutely convergent by the ratio test. The case $m=0$ can be treated quite similarly, as

$$|\varphi_1(n)|\leqslant\frac{2}{\pi\,|l|\cdot|p|}\sum_{r=0}^{\infty}\left|\sin\left(\frac{\pi pl}{2^{r+1}}\right)\cdot\prod_{j=0}^{r+1}\cos\left[\pi pl\left(1-\frac{1}{2^j}\right)\right]\right| \tag{A.33}$$

where the ratio test applied to the sine term ensures convergence.

As for the $\lim_{n\to+\infty}\varphi_2(n)$, we again distinguish the cases $m=1,2,\ldots,q-1$ and $m=0$. In the first case we have the following bound for the summand in $\varphi_2(n)$:

$$r\prod_{j=0}^{r+1}\left|\cos\left[(lq+m)\pi\frac{p}{q}\left(1-\frac{1}{2^j}\right)\right]\right| \tag{A.34}$$

which converges to zero as $r\to+\infty$, so that the existence of $\lim_{n\to+\infty}\varphi_2(n)=0$ follows from Cesàro's theorem.

For $m=0$ the limit can be rewritten in the form

$$\lim_{n\to+\infty}\frac{1}{n+1}\sum_{r=0}^{n-1}\frac{(-1)^{pl+1}\sin(\pi pl/2^{r+1})}{\pi pl(1-1/2^{r+1})}$$
$$\times\cos[(r+1)\,\omega ql]\cdot\prod_{j=0}^{r+1}\cos\left[\pi pl\left(1-\frac{1}{2^j}\right)\right] \tag{A.35}$$

and noting that the sum admits an upper bound uniform in $n\in\mathbb{N}$,

$$\frac{2}{|\pi pl|}\sum_{r=0}^{n-1} r\left|\frac{\sin(\pi pl/2^{r+1})}{\pi pl/2^{r+1}}\right|\frac{|\pi pl|}{2^{r+1}}\leqslant 2\sum_{r=0}^{\infty} r\frac{1}{2^{r+1}}<+\infty \tag{A.36}$$

we easily deduce that $\lim_{n\to+\infty}\varphi_2(n)=0$. ∎

APPENDIX B. PROOF OF PART 2 OF THE MAIN RESULT

For the existence of the limit (3.2.3) we distinguish two cases, according to the statement of part 2 of the Main Result.

Case (i). It is easy to verify that $\forall k \in \mathbb{Z}\backslash\{0\}$ we can write the identity

$$1-|Q_k|^2=4\sum_{\substack{j,j'=1\\ j'<j}}^{N} p_j p_{j'} \sin^2\frac{k\varepsilon}{2}(s_j-s_{j'}) \tag{B.1}$$

On the other hand, as $|Q_k|\leqslant 1$, we easily have

$$|Q_k-e^{-ik\omega}|\geqslant 2\sum_{\substack{j,j'=1\\ j'<j}}^{N} p_j p_{j'} \sin^2\frac{k\varepsilon}{2}(s_j-s_{j'}) \tag{B.2}$$

According to the hypothesis (2.3.2), we can write then

$$|Q_k-e^{-ik\omega}|^{-1}\leqslant \tfrac{1}{2}\gamma\,|k|^{\mu}, \qquad \forall k\in\mathbb{Z}\backslash\{0\} \tag{B.3}$$

The previous bound allows us to prove the existence of the diffusion coefficient. Indeed the expression (3.2.3) is equivalent to

$$\begin{aligned}&\lim_{n\to+\infty}\frac{n}{n+1}\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2\,\mathrm{Re}\left\{\frac{Q_k e^{ik\omega}}{1-Q_k e^{ik\omega}}\right\}\\ &\quad+\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2\,\mathrm{Re}\left\{\frac{(Q_k e^{ik\omega})^{n+2}-(Q_k e^{ik\omega})^2}{(1-Q_k e^{ik\omega})^2}\right.\\ &\quad\left.+\operatorname{cotg}\left(\frac{k\omega}{2}\right)\cdot iQ_k e^{ik\omega}\,\frac{1-(Q_k e^{ik\omega})^n}{1-Q_k e^{ik\omega}}\right\}\end{aligned} \tag{B.4}$$

and as the second series admits the upper bound

$$\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2\left[\frac{2}{|1-Q_k e^{ik\omega}|^2}+\left|\operatorname{cotg}\left(\frac{k\omega}{2}\right)\right|\frac{2}{|1-Q_k e^{ik\omega}|}\right] \tag{B.5}$$

which is convergent because of the Diophantine hypothesis on $\omega/2\pi$ and of the previous result (B.3) about $|1-Q_k e^{ik\omega}|$, we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\frac{\mathrm{E}((J_{n+1}-J_0)^2)}{2(n+1)}\\ &\quad=\frac{1}{2}\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2+\sum_{k=-\infty}^{k=+\infty}{}' |V_k|^2\,\mathrm{Re}\left\{\frac{Q_k e^{ik\omega}}{1-Q_k e^{ik\omega}}\right\} \quad\blacksquare\end{aligned} \tag{B.6}$$

There is no difficulty in proving that the condition (2.3.2) is satisfied with probability one in the sense specified by the following:

Proposition B1. The condition (2.3.2) is surely satisfied if at least one of the terms

$$\frac{\varepsilon}{2\pi}(s_j - s_{j'}), \qquad j, j' = 1, 2, ..., N \quad \text{with} \quad j > j' \tag{B.7}$$

is Diophantine. In such a case the choice of the statistical weights $p_1, ..., p_N \in]0, 1[$ is completely irrelevant.

The Lebesgue measure of the set of points $\bar{s} = ((\varepsilon/2\pi) s_1, ..., (\varepsilon/2\pi) s_N) \in \mathbb{R}^N$, such that all of the terms (B.7) are not Diophantine, is zero.

Case (ii). First we give the following lemma.

Lemma B2. The following propositions are equivalent:

(a) $\exists k \in \mathbb{Z}\backslash\{0\}$ such that $Q_k e^{ik\omega} = 1$.

(b) $\exists q_1, q_2, ..., q_N \in \mathbb{Z}$ and $k \in \mathbb{Z}\backslash\{0\}$ such that, $\forall j = 1, 2, ..., N$,

$$\varepsilon s_j = \frac{2\pi}{k} q_j - \omega \tag{B.8}$$

Proof. Let us suppose there is $k \in \mathbb{Z}\backslash\{0\}$ such that $Q_k e^{ik\omega} = 1$. By the definition of Q_k and the normalization of the measure, the condition $|Q_k|^2 = 1$ is equivalent to

$$\sum_{j, j'=1}^{N} p_j p_{j'} [\cos k\varepsilon(s_j - s_{j'}) - 1] = \sum_{\substack{j, j'=1 \\ j' < j}}^{N} p_j p_{j'} \sin^2 \frac{k\varepsilon}{2}(s_j - s_{j'}) = 0 \tag{B.9}$$

Owing to the positivity of the parameters $p_1, p_2, ..., p_N$, it is clear that the previous relationship can be satisfied if and only if

$$\sin \frac{k\varepsilon}{2}(s_j - s_{j'}) = 0, \qquad \forall j, j' = 1, 2, ..., N, \quad j' < j \tag{B.10}$$

i.e., $\forall j, j' = 1, 2, ..., N$, $j' < j$:

$$\frac{k\varepsilon}{2}(s_j - s_{j'}) = \pi m_{j,j'} \tag{B.11}$$

where $m_{j,j'} \in \mathbb{Z}\backslash\{0\}$ as, by definition, $k\varepsilon \neq 0$ and all the s_j are distinct. In this regard notice that the identity (B.11) immediately applies also to the case $j=j'$, by assuming $m_{j,j}=0$, and to $j'>j$, by setting $m_{j,j'}=-m_{j',j}$. Returning to the initial equation, we obtain

$$1=Q_k e^{ik\omega}=\sum_{j=1}^{N} p_j e^{ik\varepsilon s_j+ik\omega} \tag{B.12}$$

and by multiplying by $\exp(-ik\varepsilon s_{j'})$, with $j'\in\{1,2,\ldots,N\}$ fixed, we deduce

$$\exp(-ik\varepsilon s_{j'})=\sum_{j=1}^{N} p_j \exp[ik\varepsilon(s_j-s_{j'})+ik\omega]=\exp(ik\omega) \tag{B.13}$$

Therefore, with $k\neq 0$, there exists $q_{j'}\in\mathbb{Z}$ such that $\omega+\varepsilon s_{j'}=(2\pi/k)\,q_{j'}$, which proves the implication (a) $\Rightarrow$ (b), by the arbitrariness of $j'\in\{1,2,\ldots,N\}$.

Conversely, suppose that $\exists q_1,q_2,\ldots,q_N\in\mathbb{Z}$ and $k\in\mathbb{Z}\backslash\{0\}$ such that

$$\varepsilon s_j=\frac{2\pi}{k}q_j-\omega \qquad \forall j=1,2,\ldots,N \tag{B.14}$$

where, denoting by GCD the greatest common divisor, we can surely assume that

$$\mathrm{GCD}\{k;\mathrm{GCD}\{q_1,q_2,\ldots,q_N\}\}=1 \tag{B.15}$$

We have then,

$$\forall h\in V_\omega \stackrel{\text{def}}{=} \{h\in\mathbb{Z}\backslash\{0\}/h=kl,\, l\in\mathbb{Z}\backslash\{0\}\}$$

that

$$Q_h e^{ih\omega}=\sum_{j=1}^{N} p_j e^{ih\varepsilon s_j+ih\omega}=\sum_{j=1}^{N} p_j e^{ih(2\pi/k)\,q_j}=1 \tag{B.16}$$

Moreover, because of (B.15), the values of $h\in\mathbb{Z}\backslash\{0\}$ for which $Q_h e^{ih\omega}=1$ are only those of the set V_ω. As a conclusion, (b) $\Rightarrow$ (a) and the lemma is proved. ■

Owing to the previous result, a straightforward calculation shows that whenever the hypotheses of case (ii) occur, Eq. (3.2.5b) is satisfied. This implies the divergence of the diffusion coefficient, limit (3.2.3), which completes the proof.

ACKNOWLEDGMENTS

The work of S.S. was supported by a 60% MPI grant; the work of G.T. was partially supported by NATO grant 890383, and the work of S.V. was supported by contract CEE SC1*0281 and partially by the Ministero P.I., Italy.

The authors want to thank C. Liverani for having read the manuscript and for useful remarks.

REFERENCES

1. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
2. B. V. Chirikov, A universal instability of many-dimensional oscillator systems, *Phys. Rep.* **52**:263–379 (1979).
3. R. S. MacKay, J. D. Meiss, and I. C. Percival, Transport in Hamiltonian systems, *Physica* **13D**:55–81 (1984).
4. V. Room-Kedar and S. Wiggins, Transport on two dimensional maps, *Arch. Rat. Mech. Anal.* **109**:239 (1990).
5. I. Dana, Hamiltonian transport on unstable periodic orbits, *Physica* **39D**:205–230 (1989).
6. H. Kook and J. D. Meiss, Diffusion in symplectic maps, *Phys. Rev. A* **41**:4143–4150 (1990).
7. J. R. Cary and J. D. Meiss, Rigorously diffusive deterministic maps, *Phys. Rev.* **24A**: 2624–2668 (1981).
8. R. Artuso, G. Casati, and D. L. Shepelyansky, Breakdown of universality in renormalization dynamics for critical invariant torus, *Europhys. Lett.* **15**(4):381–386 (1991).
9. J. Bellissard and S. Vaienti, Rigorous diffusion properties for the sawtooth map, *Commun. Math. Phys.* **144**:521–536 (1992).
10. N. I. Chernov, Ergodic and statistical properties of piecewise linear hyperbolic automorphism of the 2-torus, *J. Stat. Phys.* **69**:111 (1992).
11. S. Vaienti, Ergodic properties of the discontinuous sawtooth map, *J. Stat. Phys.* **67**:251 (1992).
12. M. Wojtkowsi, A model problem with the coexistence of stochastic and integrable behaviour, *Commun. Math. Phys.* **80**:453 (1981).
13. L. A. Bunimovich and Ya. G. Sinai, Markov partitions for dispersed billiards, *Commun. Math. Phys.* **73**:247–280 (1980); Erratum, *Commun. Math. Phys.* **107**:357–358 (1986).
14. L. A. Bunimovich and Ya. G. Sinai, Statistical properties of Lorentz gas with periodic configuration of scatterers, *Commun. Math. Phys.* **78**:479 (1981).
15. L. A. Bunimovich, Ya. G. Sinai, and N. I. Chernov, Markov partitions for two dimensional hyperbolic billiards, *Russ. Math. Surv.* **45**:105–152 (1990).
16. L. A. Bunimovich, Ya. G. Sinai, and N. I. Chernow, Statistical properties of two dimensional hyperbolic billiards, *Russ. Math. Surv.* **46** (1991).
17. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer-Verlag, New York, 1975).
18. W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics* (Asterisque).
19. P. Lochak, Effective speed of Arnold diffusion and small denominators, *Phys. Lett. A* **143**:39 (1990).
20. P. Holmes and J. Marsden, Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups, *Indiana Univ. Math. J.* **32**:2 (1983).
21. B. V. Chirikov, M. A. Lieberman, D. L. Shepelyansky, and F. Vivaldi, A theory of modulational diffusion, *Physica* **14D**:289 (1985).
22. A. Bazzani, S. Siboni, G. Turchetti, and S. Vaienti, From dynamical systems to local diffusion processes, in *Chaotic Dynamics: Theory and Practice*, T. Bountis, ed. (Plenum Press, New York, 1991).
23. A. Bazzani, S. Siboni, G. Turchetti, and S. Vaienti, Diffusion in models of modulated area-preserving maps, *Phys. Rev. A* **46**:6754–6756 (1992).
24. L. Bunimovich, H. R. Jauslin, J. L. Lebowitz, A. Pellegrinotti, and P. Nielaba, Diffusive energy growth in classical and quantum driven oscillators, *J. Stat. Phys.* **62**:793–817 (1991).

25. J. R. Cary, D. F. Escande, and A. D. Verga, Nonquasilinear diffusion far from the chaotic threshold, *Phys. Rev. Lett.* **65**:3132 (1990).
26. A. Bazzani, S. Siboni, G. Turchetti, and S. Vaienti, A model of modulated diffusion. II. Numerical results on statistical properties, *J. Stat. Phys.*, this issue.
27. A. D. Bruno, The analytical form of differential equation, *Trans. Mosc. Math. Soc.* **25**:131–288 (1971); **26**:199–239 (1972).
28. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, New York, 1982).
29. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
30. S. Siboni, Some ergodic properties of a mapping obtained by coupling the translation of the 1-torus with the endomorphism $2x \bmod [0, 1[$, Preprint, Centre de Physique Théorique du CNRS, Luminy, Marseille, France, CPT-91/P.2690 (1992); *Nonlinearity*, in press.
31. K. Petersen, *Ergodic Theory* (Cambridge University Press, Cambridge, 1983).
32. L. M. Abramov and V. A. Rokhlin, The entropy of a skew product of measure-preserving transformations, *Trans. Am. Math. Soc. Ser. 2* **48**:225–265 (1965).
33. H. Anzai, Ergodic skew-product transformations on the torus, *Osaka Math. J.* **1**:83–99 (1951).
34. E. Lukacs, *Characteristic Functions* (Griffin, London, 1970).
35. S. Siboni, Rilassamento all'equilibrio in un sistema mixing e analisi di un modello di diffusione modulata, Tesi di Dottorato di Ricerca dell'Università degli Studi di Bologna, Italy (1991).
36. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1965).